Duality in Logic, Games and Categories

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Duality Theory

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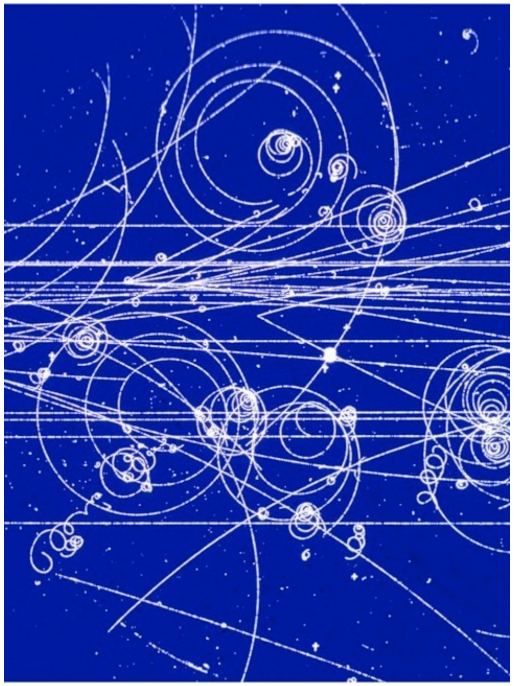
Logic

Physics



What are the symmetries of logic?

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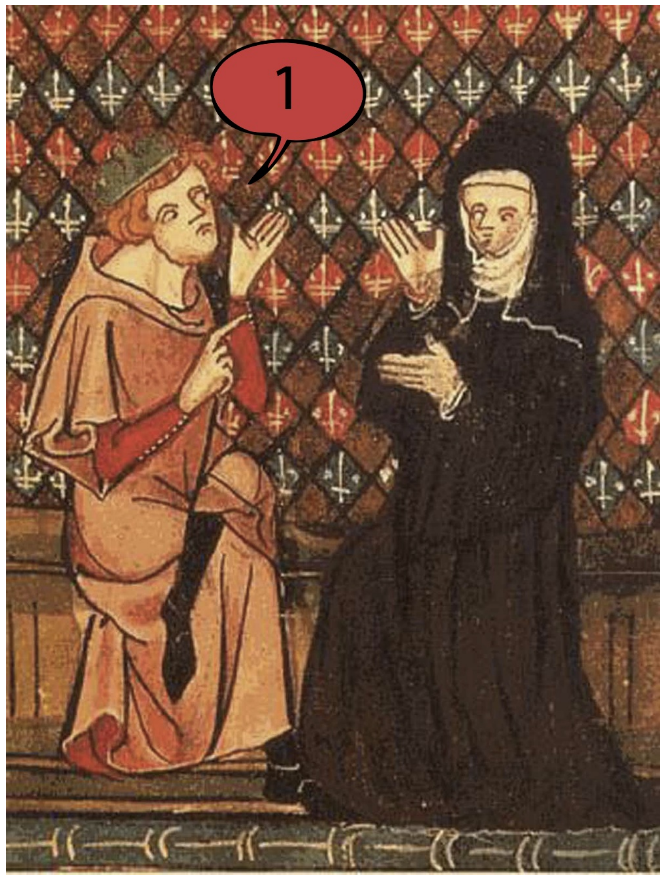

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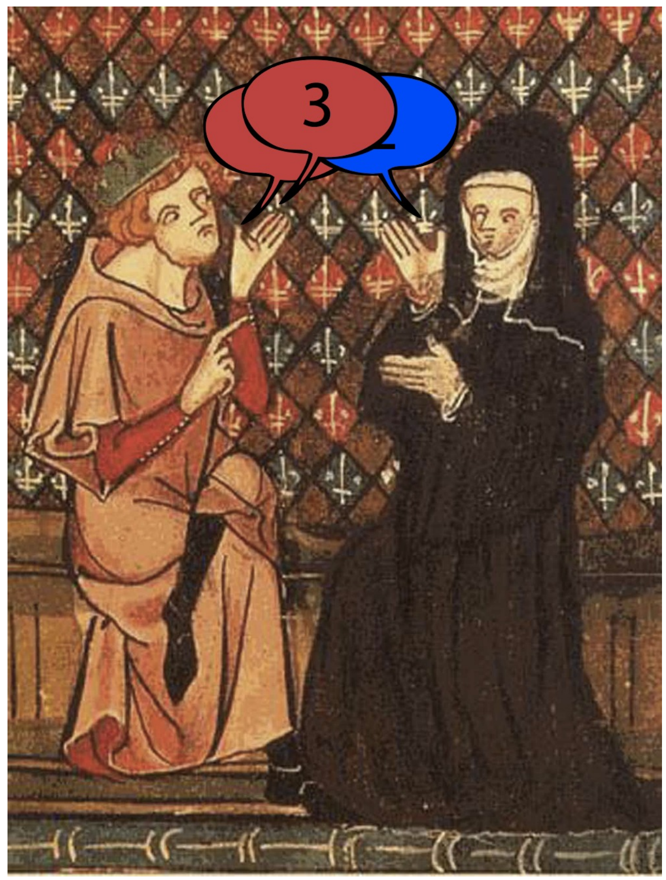
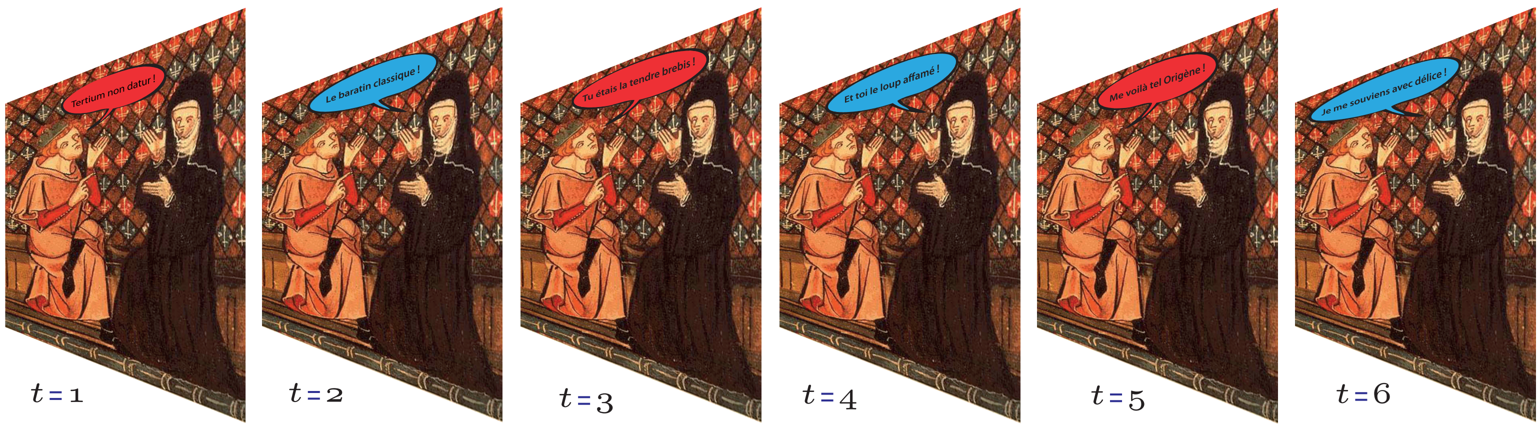


A logical space-time



Emerges in the semantics of low level languages

The basic symmetry of logic

The logical discourse is **symmetric** between Player and Opponent

Claim: this symmetry is the foundation of logic

So, what can we learn from this basic symmetry?

De Morgan duality

The duality relates the conjunction and the disjunction of classical logic:

$$\boxed{\begin{array}{ccc} (A \vee B)^* & \cong & B^* \wedge A^* \\ (A \wedge B)^* & \cong & B^* \vee A^* \end{array}}$$

De Morgan duality in a constructive scenario

Can we make sense of this involutive negation

$$\boxed{A^{**} \quad \cong \quad A}$$

in a constructive logic like intuitionistic logic?

In particular, can we decompose the intuitionistic implication as

$$\boxed{A \Rightarrow B \quad \cong \quad A^{*} \ \vee \ B}$$

Guideline: game semantics

Every proof of formula A initiates a dialogue where

Proponent tries to convince Opponent

Opponent tries to refute Proponent

An interactive approach to logic and programming languages

The formal proof of the drinker's formula

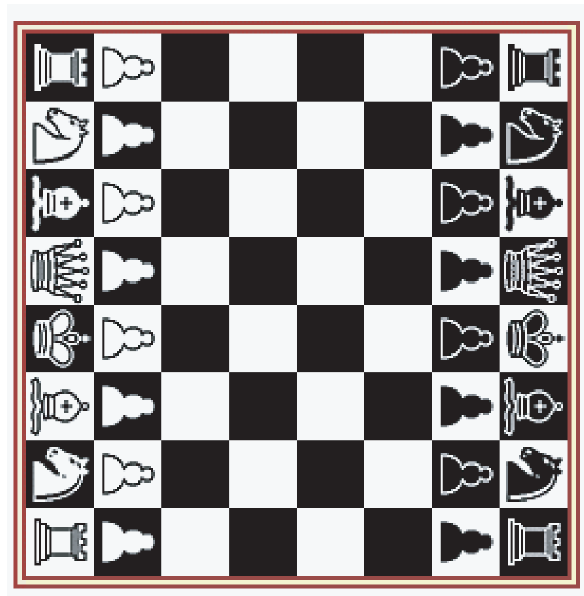
$$
\begin{array}{cl}
\hline & \text{Axiom} \\
A(x_0) \vdash A(x_0) & \\
\hline & \text{Right Weakening} \\
A(x_0) \vdash A(x_0), \forall x.A(x) & \\
\hline & \text{Right} \Rightarrow \\
\vdash A(x_0), A(x_0) \Rightarrow \forall x.A(x) & \\
\hline & \text{Right } \exists \\
\vdash A(x_0), \exists y.\{A(y) \Rightarrow \forall x.A(x)\} & \\
\hline & \text{Right } \forall \\
\vdash \forall x.A(x), \exists y.\{A(y) \Rightarrow \forall x.A(x)\} & \\
\hline & \text{Left Weakening} \\
A(y_0) \vdash \forall x.A(x), \exists y.\{A(y) \Rightarrow \forall x.A(x)\} & \\
\hline & \text{Right} \Rightarrow \\
\vdash A(y_0) \Rightarrow \forall x.A(x), \exists y.\{A(y) \Rightarrow \forall x.A(x)\} & \\
\hline & \text{Right } \exists \\
\vdash \exists y.\{A(y) \Rightarrow \forall x.A(x)\}, \exists y.\{A(y) \Rightarrow \forall x.A(x)\} & \\
\hline & \text{Contraction} \\
\vdash \exists y.\{A(y) \Rightarrow \forall x.A(x)\} &
\end{array}
$$

Duality

Proponent
Program

plays the game

A



Opponent
Environment

plays the game

$\neg A$

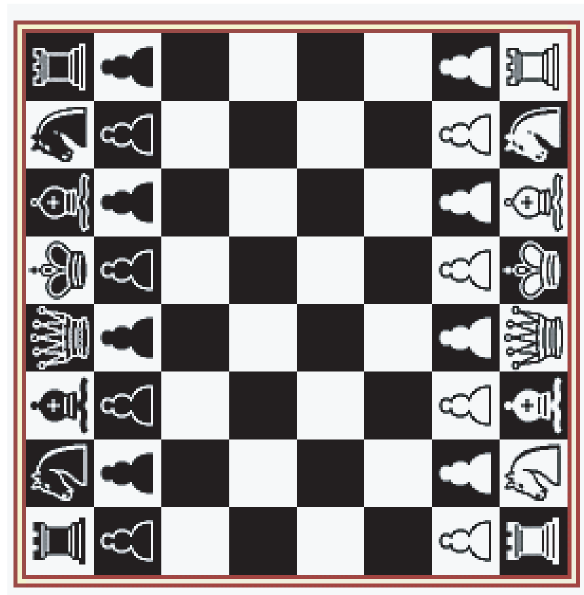
Negation permutes the rôles of Proponent and Opponent

Duality

Opponent
Environment

plays the game

$\neg A$



Proponent
Program

plays the game

A

Negation permutes the rôles of Opponent and Proponent

Classical duality in a boolean algebra

Negation defines a **bijection**

$$\begin{tikzcd} B \arrow[rr, bend left, "negation"] \arrow[rr, phantom, "\perp"] & & B^{op} \arrow[ll, bend left, "negation"] \end{tikzcd}$$

between the boolean algebra B and its opposite boolean algebra B^{op}.

Intuitionistic negation in a Heyting algebra

Every object $\perp$ defines a **Galois connection**

$$\begin{tikzcd} H \arrow[rr, bend left, "negation"] \arrow[rr, phantom, "\perp"] & & H^{op} \arrow[ll, bend left, "negation"] \end{tikzcd}$$

between the Heyting algebra H and its opposite algebra H^{op}.

$$a \leqslant_H \perp \multimapinv b \qquad \Longleftrightarrow \qquad b \leqslant_H a \multimap \perp \qquad \Longleftrightarrow \qquad a \multimap \perp \leqslant_{H^{op}} b$$

Double negation translation

Every object $\perp$ defines a **Galois connection**

$$\begin{tikzcd} H \arrow[rr, bend left, "negation"] \arrow[rr, phantom, "\perp"] & & H^{op} \arrow[ll, bend left, "negation"] \end{tikzcd}$$

between the Heyting algebra H and its opposite algebra H^{op}.

The negated elements of a Heyting algebra form a Boolean algebra.

The functorial approach to proof invariants

Cartesian closed categories

Cartesian closed categories

A **cartesian** category $\mathscr{C}$ is **closed** when there exists a functor

$$\Rightarrow \quad : \quad \mathscr{C}^{op} \times \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

and a natural bijection

$$\varphi_{A,B,C} \quad : \quad \mathscr{C}(A \times B, C) \cong \mathscr{C}(B, A \Rightarrow C)$$

The free cartesian closed category

The objects of the category **free-ccc**$(\mathscr{C})$ are the formulas

$$A, B \quad ::= \quad X \quad | \quad A \times B \quad | \quad A \Rightarrow B \quad | \quad \mathit{1}$$

where X is an object of the category $\mathscr{C}$.

The morphisms are the simply-typed λ-terms, modulo $\beta\eta$-conversion.

In particular, the $\beta\eta$-normal forms provide a "basis" of the free ccc.

The simply-typed λ-calculus

Variable

$$\frac{}{x : A \vdash x : A}$$

Abstraction

$$\frac{\Gamma, x : A \vdash P : B}{\Gamma \vdash \lambda x.P : A \Rightarrow B}$$

Application

$$\frac{\Gamma \vdash P : A \Rightarrow B \qquad \Delta \vdash Q : A}{\Gamma, \Delta \vdash PQ : B}$$

Weakening

$$\frac{\Gamma \vdash P : B}{\Gamma, x : A \vdash P : B}$$

Contraction

$$\frac{\Gamma, x : A, y : A \vdash P : B}{\Gamma, z : A \vdash P[x, y \leftarrow z] : B}$$

Exchange

$$\frac{\Gamma, x : A, y : B, \Delta \vdash P : C}{\Gamma, y : B, x : A, \Delta \vdash P : C}$$

The simply-typed λ-calculus [with products]

Pairing
$$\frac{\Gamma \vdash P : A \qquad \Gamma \vdash Q : B}{\Gamma \vdash \langle P, Q \rangle : A \times B}$$

Left projection
$$\frac{\Gamma \vdash P : A \times B}{\Gamma \vdash \pi_1\, P : A}$$

Right projection
$$\frac{\Gamma \vdash P : A \times B}{\Gamma \vdash \pi_2\, P : B}$$

Unit
$$\frac{}{\Gamma \vdash * : 1}$$

Execution of λ-terms

In order to compute a λ-term, one applies the β-rule

$$\boxed{(\lambda x.P)\, Q \;\longrightarrow_\beta\; P\,[x := Q]}$$

which substitutes the argument Q for every instance of the variable x in the body P of the function. One may also apply the η-rule:

$$\boxed{P \;\longrightarrow_\eta\; \lambda x.\,(Px)}$$

Proof invariants

Every ccc $\mathscr{D}$ induces a proof invariant $[-]$ modulo execution

$$\begin{tikzcd} \textbf{free-ccc}(\mathscr{C}) \arrow[rr, bend left, "{[-]}"] & & \mathscr{D} \\ & \mathscr{C} \arrow[ul, "\textit{atoms}"] \arrow[ur, "\textit{interpretation of atoms}"'] & \end{tikzcd}$$

A purely syntactic and type-theoretic construction

An apparent obstruction to duality

Self-duality in cartesian closed categories

Duality in a boolean algebra

Negation defines a **bijection**

$$\begin{tikzcd} B \arrow[rr, bend left, "negation"] \arrow[rr, phantom, "\perp"] & & B^{op} \arrow[ll, bend left, "negation"] \end{tikzcd}$$

between the boolean algebra B and its opposite boolean algebra B^{op}.

Duality in a category

One would like to think that negation defines an **equivalence**

$$\begin{tikzcd} \mathscr{C} \arrow[rr, bend left, "negation"] \arrow[rr, phantom, "\perp"] & & \mathscr{C}^{op} \arrow[ll, bend left, "negation"] \end{tikzcd}$$

between a cartesian closed category $\mathscr{C}$ and its opposite category $\mathscr{C}^{op}$.

However, in a cartesian closed category...

Suppose that the category $\mathscr{C}$ has an initial object 0. Then,

$$\boxed{\text{Every object } A \times 0 \text{ is also initial.}}$$

The reason is that

$$\mathscr{C}(A \times 0, B) \quad \cong \quad \mathscr{C}(0, A \Rightarrow B) \quad \cong \quad singleton$$

for every object B of the category $\mathscr{C}$.

However, in a cartesian closed category...

Suppose that the category $\mathscr{C}$ has an initial object 0. Then,

$$\boxed{\text{Every object } A \times 0 \text{ is initial... and thus isomorphic to } 0.}$$

The reason is that

$$\mathscr{C}(A \times 0, B) \quad \cong \quad \mathscr{C}(0, A \Rightarrow B) \quad \cong \quad \mathit{singleton}$$

for every object B of the category $\mathscr{C}$.

However, in a cartesian closed category...

Every morphism $f : A \longrightarrow 0$ is an isomorphism.

Given such a morphism $f : A \to 0$, consider the morphism $h : A \to A \times 0$ making the diagram commute:

$$
\begin{tikzcd}
 & A \arrow[ddl, bend right, "id"'] \arrow[d, dashed, "h"] \arrow[ddr, bend left, "f"] & \\
 & A \times 0 \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
A & & 0
\end{tikzcd}
$$

In a self-dual cartesian closed category...

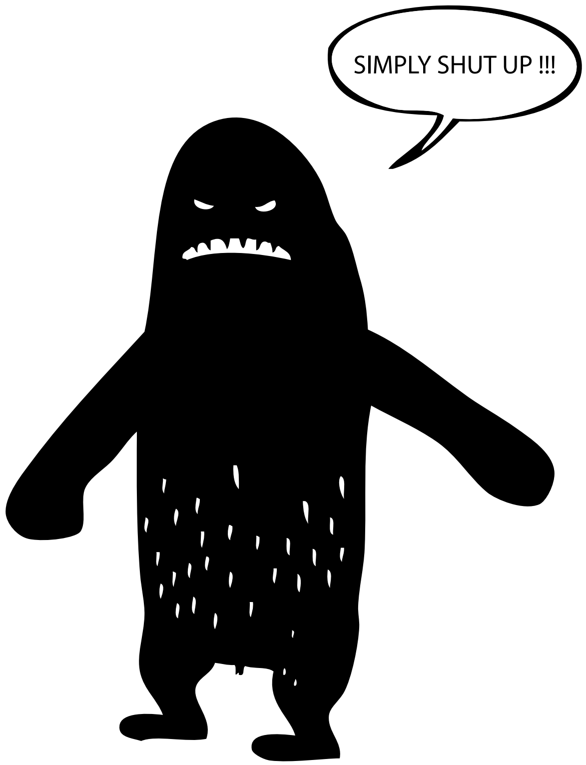
$$
\begin{aligned}
Hom(\ A\ ,\ B\ ) \quad &\cong \quad Hom(\ A \times 1\ ,\ B\ ) \\
&\cong \quad Hom(\ 1\ ,\ A \Rightarrow B\ ) \\
&\cong \quad Hom(\ \neg(A \Rightarrow B)\ ,\ \neg 1\ ) \\
&\cong \quad Hom(\ \neg(A \Rightarrow B)\ ,\ 0\ ) \\
&\cong \quad \text{empty or singleton}
\end{aligned}
$$

Hence, every such self-dual category $\mathscr{C}$ is a preorder !

The microcosm principle

An idea coming from higher-dimensional algebra

The microcosm principle



No contradiction (thus no formal logic) can emerge in a tyranny...

A microcosm principle in algebra [Baez & Dolan 1997]

The definition of a **monoid**

$$M \times M \longrightarrow M$$

requires the ability to define a **cartesian product** of sets

$$A \, , \, B \mapsto A \times B$$

Structure at dimension 0 requires structure at dimension 1

A microcosm principle in algebra
[Baez & Dolan 1997]

The definition of a **cartesian** category

$$\mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$$

requires the ability to define a **cartesian product** of categories

$$\mathscr{A}\ ,\ \mathscr{B} \longmapsto \mathscr{A} \times \mathscr{B}$$

Structure at dimension 1 requires structure at dimension 2

A similar microcosm principle in logic

The definition of a cartesian **closed** category

$$\mathscr{C}^{op} \quad \times \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

requires the ability to define the **opposite** of a category

$$\mathscr{A} \quad \mapsto \quad \mathscr{A}^{op}$$

Hence, the “implication” at level 1 requires a “negation” at level 2

An automorphism in $\underline{Cat}$

The 2-functor

$$op \quad : \quad \underline{Cat} \longrightarrow \underline{Cat}^{\,op(2)}$$

transports every natural transformation

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathscr{D} \arrow[Rightarrow, from=U, to=D, "\theta"] \end{tikzcd}$$

to a natural transformation in the opposite direction:

$$\begin{tikzcd} \mathscr{C}^{\,op} \arrow[r, bend left, "F^{\,op}"{name=U}] \arrow[r, bend right, "G^{\,op}"'{name=D}] & \mathscr{D}^{\,op} \arrow[Rightarrow, from=D, to=U, "\theta^{\,op}"'] \end{tikzcd}$$

$\longrightarrow$ requires a braiding on $\mathscr{V}$ in the case of $\mathscr{V}$-enriched categories

Chiralities

A bilateral account of categories

From categories to chiralities

This leads to a slightly bizarre idea:

decorrelate the category $\mathscr{C}$ from its opposite category $\mathscr{C}^{op}$

So, let us define a **chirality** as a pair of categories $(\mathscr{A}, \mathscr{B})$ such that

$$\mathscr{A} \cong \mathscr{C} \qquad\qquad \mathscr{B} \cong \mathscr{C}^{op}$$

for some category $\mathscr{C}$.

Here $\cong$ means **equivalence** of category

Chirality

More formally:

Definition:

A chirality is a pair of categories $(\mathscr{A}, \mathscr{B})$ equipped with an equivalence:

$$\begin{tikzcd} \mathscr{A} \arrow[rr, bend left, "(-)^*"] \arrow[rr, phantom, "equivalence"] & & \mathscr{B}^{op} \arrow[ll, bend left, "{}^*(-)"] \end{tikzcd}$$

A 2-categorical justification

Let $\mathcal{Chir}$ denote the 2-category with

- chiralities as objects
- chirality homomorphism as 1-dimensional cells
- chirality transformations as 2-dimensional cells

Proposition. The 2-category $\mathcal{Chir}$ is biequivalent to the 2-category $\mathcal{Cat}$.

Cartesian closed chiralities

A 2-sided account of cartesian closed categories

Cartesian chiralities

Definition. A cartesian chirality is a chirality

- ▷ whose category $\mathscr{A}$ has **finite products** noted

$$a_1 \wedge a_2 \qquad\qquad true$$

- ▷ whose category $\mathscr{B}$ has **finite sums** noted

$$b_1 \vee b_2 \qquad\qquad false$$

Cartesian closed chiralities

Definition. A cartesian closed chirality is a cartesian chirality

$$(\mathscr{A}, \wedge, true) \qquad\qquad (\mathscr{B}, \vee, false)$$

equipped with a pseudo-action

$$\vee \quad : \quad \mathscr{B} \quad \times \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

and a bijection

$$\mathscr{A}\,(\,a_1 \;\wedge\; a_2\,,\; a_3\,) \quad\cong\quad \mathscr{A}(\,a_1\,,\; a_2^* \;\vee\; a_3\,)$$

natural in a_1, a_2 and a_3.

Dictionary

The pseudo-action

$$\vee \quad : \quad \mathscr{B} \quad \times \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

reflects the implication

$$\textit{implies} \quad : \quad \mathscr{C}^{op} \quad \times \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

Dictionary

The isomorphism of the pseudo-action

$$(b_1 \ \vee \ b_2) \ \vee \ a \quad \cong \quad b_1 \ \vee \ (b_2 \ \vee \ a)$$

reflects the familiar isomorphism

$$(x_1 \ \mathit{and} \ x_2) \ \mathit{implies} \ y \quad \cong \quad x_1 \ \mathit{implies} \ (x_2 \ \mathit{implies} \ y)$$

of cartesian closed categories.

Dictionary continued

The isomorphism

$$\mathscr{A}(a_1 \wedge a_2, a_3) \quad \cong \quad \mathscr{A}(a_2, a_1^* \vee a_3)$$

reflects the familiar isomorphism

$$\mathscr{A}(x \; \mathit{and} \; y, z) \quad \cong \quad \mathscr{A}(y, x \; \mathit{implies} \; z)$$

of cartesian closed categories.

Key observation

The isomorphism

$$\boxed{a_1 \; \textit{implies} \; a_2 \quad \cong \quad a_1^* \; \vee \; a_2}$$

deserves the name of

« classical decomposition of the implication »

although we work here in a cartesian closed category...

Key observation

This means that the decomposition

$$\boxed{a_1 \ \textit{implies} \ a_2 \qquad \cong \qquad a_1^* \ \vee \ a_2}$$

is a **principle of logic** which comes from the **2-dimensional duality**

$$\mathscr{C} \quad \mapsto \quad \mathscr{C}^{op}$$

rather than from the **1-dimensional duality**

$$A \quad \mapsto \quad A^*$$

specific to classical logic or to linear logic.

Isbell duality compared to Dedekind-MacNeille completion

A comparison between orders and categories

Ideal completion

Every partial order A generates a free complete $\bigvee$-lattice $\widehat{A}$

$$A \longrightarrow \widehat{A}$$

whose elements are the downward closed subsets of A, with

$$\varphi \leqslant_{\widehat{A}} \psi \iff \varphi \subseteq \psi.$$

$$\boxed{\widehat{A} = A^{op} \Rightarrow \{0,1\}}$$

Free colimit completions of categories

Every small category $\mathscr{C}$ generates a free cocomplete category $\mathscr{P}\mathscr{C}$

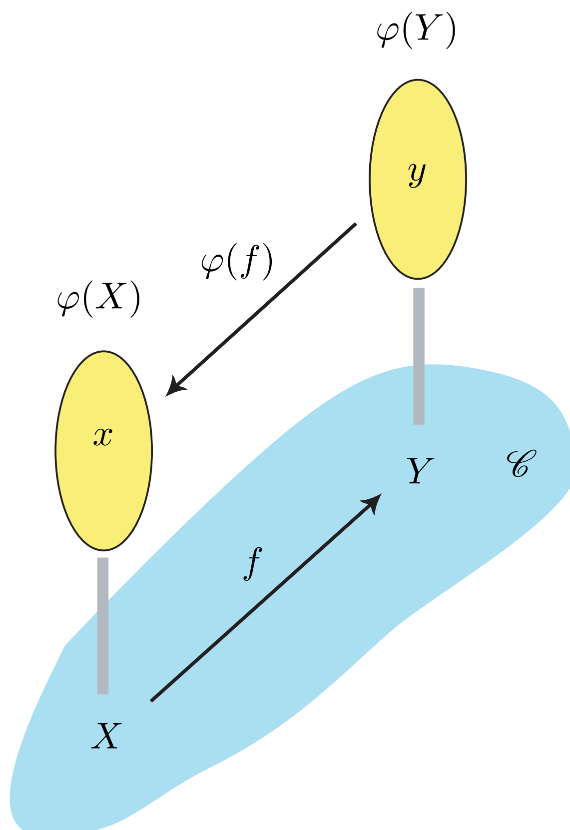
$$\mathscr{C} \quad \longrightarrow \quad \mathscr{P}\mathscr{C}$$

whose elements are the presheaves over $\mathscr{C}$, with

$$\varphi \quad \longrightarrow_{\mathscr{P}\mathscr{C}} \quad \psi \qquad \Longleftrightarrow \qquad \varphi \quad \xrightarrow{natural} \quad \psi.$$

$$\boxed{\mathscr{P}\mathscr{C} \quad = \quad \mathscr{C}^{op} \Rightarrow Set}$$

Contravariant presheaves



Replace downward closed sets

Filter completion

Every partial order A generates a free complete $\bigwedge$-lattice $\check{A}$

$$A \longrightarrow \check{A}$$

whose elements are the upward closed subsets of A, with

$$\varphi \quad \leqslant_{\check{A}} \quad \psi \qquad \Longleftrightarrow \qquad \varphi \quad \supseteq \quad \psi.$$

$$\boxed{\check{A} \quad = \quad (\, A \Rightarrow \{0,1\}\,)^{op}}$$

Free limit completions of categories

Every small category $\mathscr{C}$ generates a free complete category $\mathscr{Q}\mathscr{C}$

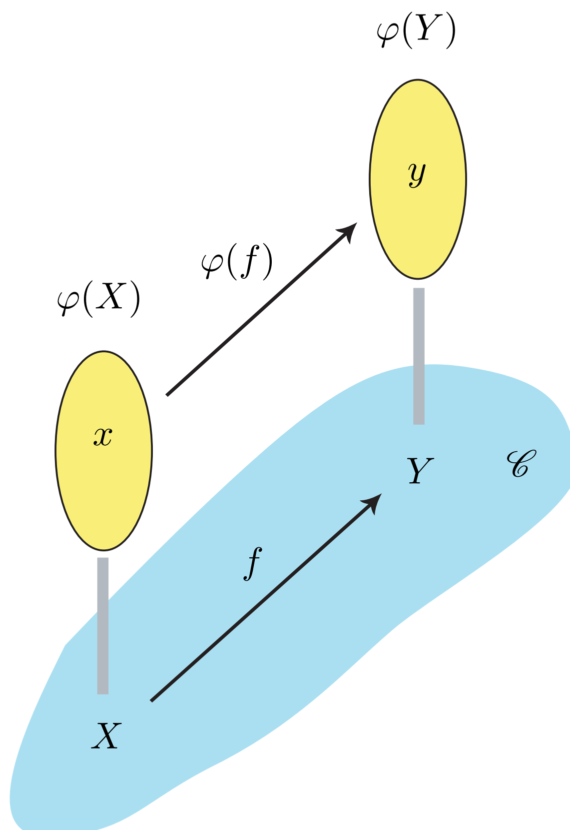
$$\mathscr{C} \quad \longrightarrow \quad \mathscr{Q}\mathscr{C}$$

whose elements are the covariant presheaves over $\mathscr{C}$, with

$$\varphi \quad \longrightarrow_{\mathscr{Q}\mathscr{C}} \quad \psi \qquad \Longleftrightarrow \qquad \varphi \quad \xleftarrow{natural} \quad \psi.$$

$$\boxed{\mathscr{Q}\mathscr{C} \quad = \quad \mathscr{P}(\mathscr{C}^{op})^{op} \quad = \quad (\mathscr{C} \Rightarrow Set)^{op}}$$

Covariant presheaves



Replaces upward closed sets

The Dedekind-MacNeille completion

A Galois connection

$$\begin{tikzcd} \widehat{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \check{A} \arrow[l, bend left, "R"] \end{tikzcd}$$

$$\boxed{\begin{aligned} L(\varphi) &= \{\ y \mid \forall x \in \varphi,\ \ x \leqslant_A y\ \} \\ R(\psi) &= \{\ x \mid \forall y \in \psi,\ \ x \leqslant_A y\ \} \end{aligned}}$$

$$\varphi \subseteq R(\psi) \quad\Longleftrightarrow\quad \forall x \in \varphi, y \in \psi,\ x \leqslant_A y \quad\Longleftrightarrow\quad L(\varphi) \supseteq \psi$$

The completion keeps the pairs (φ, ψ) such that $\psi = L(\varphi)$ and $\varphi = R(\psi)$

The Isbell conjugation

One obtains the adjunction

$$\begin{tikzcd} \mathscr{P}\mathscr{C} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathscr{Q}\mathscr{C} \arrow[ll, bend left, "R"] \end{tikzcd}$$

$$\boxed{\begin{array}{rcccccl} L(\varphi) & : & Y & \mapsto & \mathscr{P}\mathscr{C}(\varphi, Y) & = & \int_{X \in \mathscr{C}} \varphi(X) \Rightarrow hom(X, Y) \\ R(\psi) & : & X & \mapsto & \mathscr{Q}\mathscr{C}(X, \psi) & = & \int_{Y \in \mathscr{C}} \psi(Y) \Rightarrow hom(X, Y) \end{array}}$$

The Isbell conjugation

The adjunction

$$\begin{tikzcd} \mathscr{P}\mathscr{C} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathscr{Q}\mathscr{C} \arrow[ll, bend left, "R"] \end{tikzcd}$$

comes from the natural bijections

$$\mathscr{P}\mathscr{C}(\varphi, R(\psi)) \;\cong\; \int_{X,Y\in\mathscr{C}} \varphi(X) \times \psi(Y) \Rightarrow hom(X,Y) \;\cong\; \mathscr{Q}\mathscr{C}(L(\varphi), \psi)$$

The Isbell conjugation

Proposition. Suppose given a contravariant presheaf

$$\varphi \quad : \quad \mathscr{C}^{op} \quad \longrightarrow \quad Set$$

which defines a small colimit in the original category $\mathscr{C}$.

In that case, $L(\varphi)$ is representable and

$$R \circ L(\varphi) \quad \cong \quad colim\,\varphi.$$

Unfortunately, the Isbell envelope does not have limit nor colimits...

Back to the fundamental symmetry

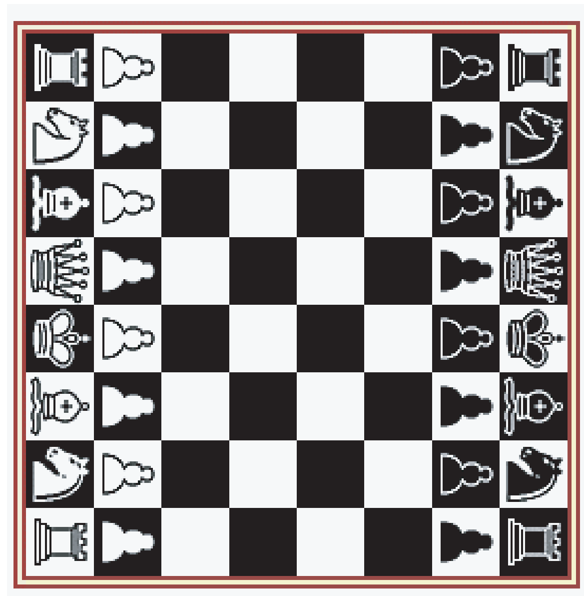
What does the chirality tell us about games?

Duality

Proponent
Program

plays the game

A



Opponent
Environment

plays the game

$\neg A$

Negation permutes the rôles of Proponent and Opponent

Duality

Opponent
Environment

plays the game

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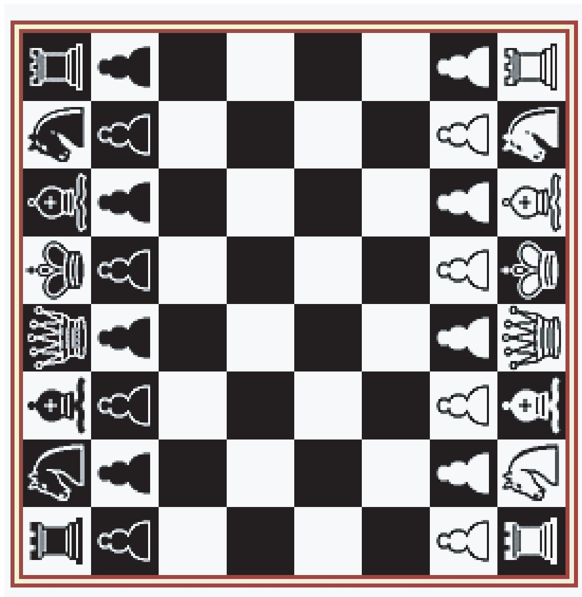


Proponent
Program

plays the game

A

Negation permutes the rôles of Opponent and Proponent

Tensor product

$\otimes$

Player and Opponent play the two games in parallel

Sum

$\oplus$

Proponent selects one component

Product

&

Opponent selects one component

Exponentials

 $\otimes$  $\otimes$  $\otimes$ $\cdots$

Opponent opens as many copies as necessary to beat Proponent

The category of simple games

An idea dating back to André Joyal in 1977

Simple games

A simple game (M, P, λ) consists of

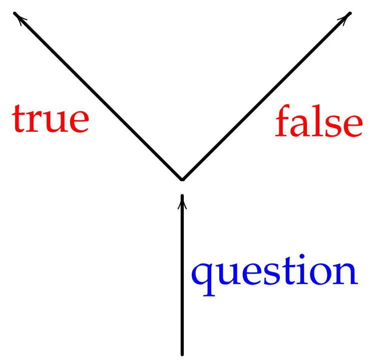
M	a finite set of **moves**,
$P \subseteq M^*$	a set of **plays**,
$\lambda : M \to \{-1, +1\}$	a **polarity** function on moves

such that every play is **alternating** and **starts by Opponent.**

Alternatively, a simple game is an alternating decision tree.

Simple games

The boolean game $\mathbb{B}$:



Player in red
Opponent in blue

Deterministic strategies

A **strategy** σ is a set of **alternating plays** of **even-length**

$$s \quad = \quad m_1 \cdots m_{2k}$$

such that:

– σ contains **the empty play,**

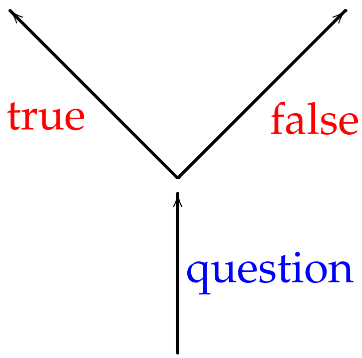
– σ is **closed by even-length prefix:**

$$\forall s, \forall m, n \in M, \qquad s \cdot m \cdot n \in \sigma \;\Rightarrow\; s \in \sigma$$

– σ is **deterministic:**

$$\forall s \in \sigma, \forall m, n_1, n_2 \in M, \qquad s \cdot m \cdot n_1 \in \sigma \text{ and } s \cdot m \cdot n_2 \in \sigma \;\Rightarrow\; n_1 = n_2.$$

Three strategies on the boolean game $\mathbb{B}$



Player in red
Opponent in blue

Total strategies

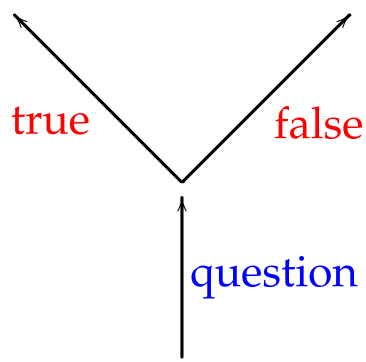
A strategy σ is **total** when

– for every play s of the strategy σ

– for every Opponent move m such that $s \cdot m$ is a play

there exists a Proponent move n such that $s \cdot m \cdot n$ is a play of σ.

Two total strategies on the boolean game $\mathbb{B}$



Player in red
Opponent in blue

Tensor product

Given two simple games A and B, define

$$A \otimes B$$

as the simple game

$$M_{A \otimes B} = M_A + M_B$$

$$\lambda_{A \otimes B} = [\lambda_A, \lambda_B]$$

$$P_{A \otimes B} = P_A \otimes P_B$$

where $P_A \otimes P_B$ denotes the set of alternating plays in $M^*_{A \otimes B}$ obtained by interleaving a play $s \in P_A$ and a play $t \in P_B$.

Linear implication

Given two simple games A and B, define

$$A \quad \multimap \quad B$$

as the simple game

$$M_{A \multimap B} \quad = \quad M_A \; + \; M_B$$

$$\lambda_{A \multimap B} \quad = \quad [\; -\lambda_A \;,\; \lambda_B \;]$$

$$P_{A \multimap B} \quad = \quad P_A \; \multimap \; P_B$$

where $P_A \multimap P_B$ denotes the set of alternating plays in $M^*_{A \multimap B}$ obtained by interleaving a play $s \in P_A$ and a play $t \in P_B$.

A category of simple games

The category Games has

– the simple games as objects

– the total strategies of the simple game

$$\sigma \quad \in \quad A \quad \multimap \quad B$$

as maps

$$\sigma \quad : \quad A \quad \longrightarrow \quad B$$

The copycat strategy

The identity map

$$id_A \quad : \quad A \quad \longrightarrow \quad A$$

is the **copycat** strategy

$$id_A \quad : \quad A \quad \multimap \quad A$$

defined as

$$id_A \quad = \quad \{ \quad s \ \in \ P_{A \multimap A} \quad | \quad s \ = \ m_1 \cdot m_1 \cdot m_2 \cdot m_2 \ \cdots \ m_k \cdot m_k \quad \}$$

The copycat strategy

$$
\begin{array}{ccc}
A & \xrightarrow{id} & A \\
 & & m_1 \\
m_1 & & \\
m_2 & & \\
 & & m_2 \\
 & & m_3 \\
m_3 & & \\
\vdots & & \vdots \\
m_k & & \\
 & & m_k
\end{array}
$$

Composition

Given two strategies

$$A \xrightarrow{\sigma} B \xrightarrow{\tau} C$$

the composite strategy

$$A \xrightarrow{\sigma\,;\,\tau} C$$

is defined as

$$\sigma\,;\,\tau \quad = \quad \left\{ \; u \in P_{A \multimap C} \;\middle|\; \exists\, s \in \sigma,\, \exists\, t \in \tau \quad \begin{array}{c} u_{\upharpoonright A} = s_{\upharpoonright A} \\ s_{\upharpoonright B} = t_{\upharpoonright B} \\ u_{\upharpoonright C} = t_{\upharpoonright C} \end{array} \right\}$$

The definition of composition is associative.

Illustration

$$1 \xrightarrow{\mathrm{true}} \mathbb{B} \xrightarrow{id} \mathbb{B}$$

$$\mathrm{q} \qquad\qquad \mathrm{q}$$

$$\mathrm{true} \qquad\qquad \mathrm{true}$$

Illustration

$$1 \xrightarrow{\text{false}} \mathbb{B} \xrightarrow{id} \mathbb{B}$$

$$\begin{array}{ccc} & & \mathrm{q} \\ & \mathrm{q} & \\ & \text{false} & \\ & & \text{false} \end{array}$$

Illustration

$$1 \xrightarrow{\text{true}} \mathbb{B} \xrightarrow{\text{negation}} \mathbb{B}$$

$$\begin{array}{ccc} & & \text{q} \\ & \text{q} & \\ & \text{true} & \\ & & \text{false} \end{array}$$

An important isomorphism

The simple game

$$(\quad A \quad \otimes \quad B \quad) \quad \multimap \quad C$$

is isomorphic to the simple game

$$A \quad \multimap \quad (\quad B \quad \multimap \quad C \quad)$$

for all simple games A, B, C.

Here, isomorphism means tree isomorphism.

The category of simple games

Theorem.

The category *Games* is symmetric monoidal closed.

As such, it defines a model of the linear λ-calculus.

Illustration

$$
\begin{array}{ccccccc}
(\mathbb{B} & \multimap & \mathbb{B}) & \otimes & \mathbb{B} & \xrightarrow{\text{eval}} & \mathbb{B} \\
 & & & & & & \mathrm{q} \\
 & & & & \color{red}{\mathrm{q}} & & \\
 & & & & \mathrm{bool} & & \\
 & & & & & & \color{red}{\mathrm{bool}}
\end{array}
$$

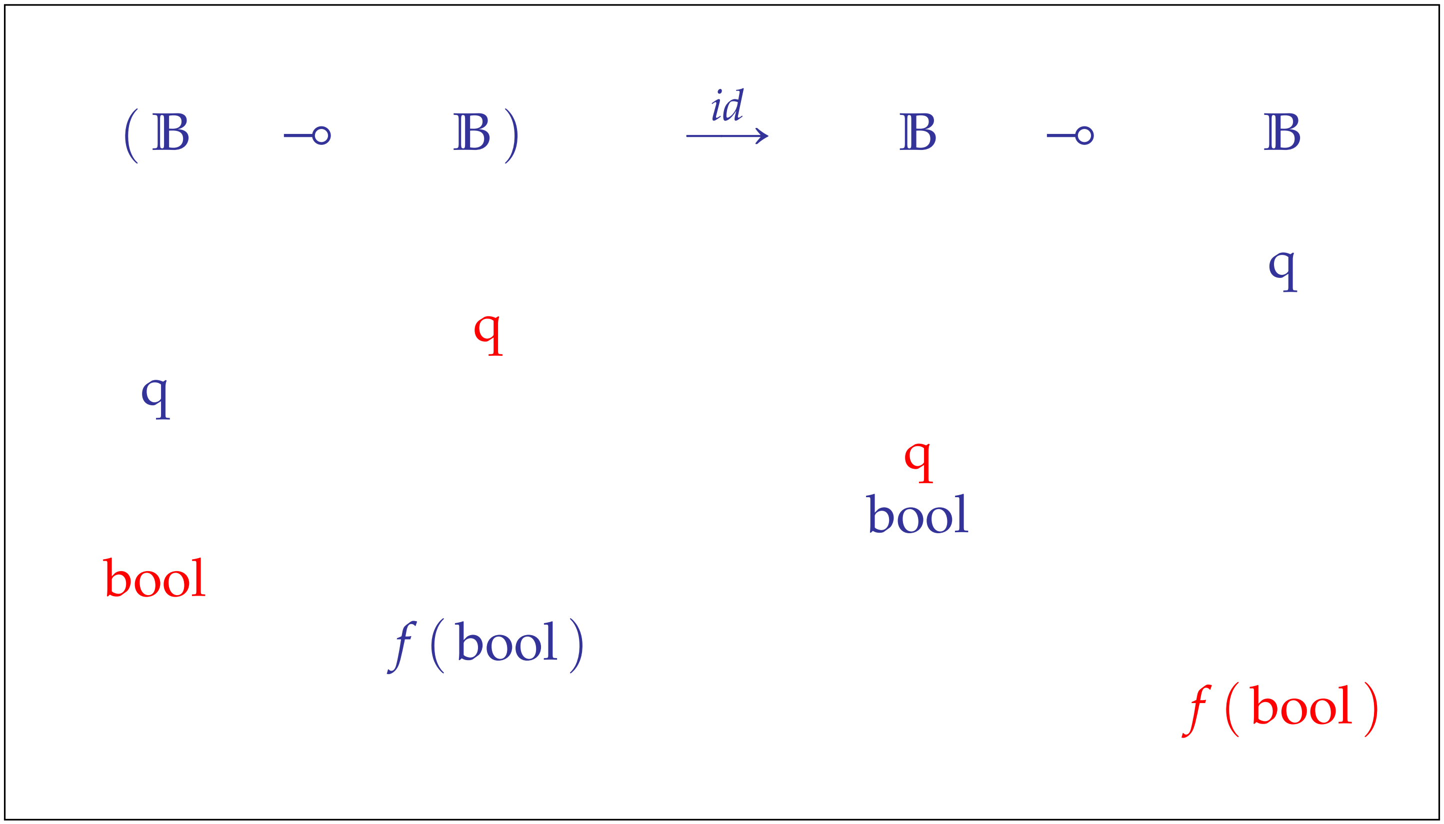
$$f : \mathbb{B} \multimap \mathbb{B} \;,\; x : \mathbb{B} \;\vdash\; f(x) : \mathbb{B}$$

Illustration

$(\mathbb{B}$	$\multimap$	$\mathbb{B})$	$\otimes$	$\mathbb{B}$	$\xrightarrow{\text{eval}}$	$\mathbb{B}$
						q
		q				
q						
				q		
				bool		
bool						
		$f(\text{bool})$				
						$f(\text{bool})$

$$f : \mathbb{B} \multimap \mathbb{B} \;,\; x : \mathbb{B} \;\vdash\; f(x) : \mathbb{B}$$

Illustration



$$f : \mathbb{B} \multimap \mathbb{B} \quad \vdash \quad \lambda x \,.\, f(x) : \mathbb{B} \multimap \mathbb{B}$$

Currification

More generally, the transformation of the term

$$\Gamma \;,\; x : A \;\vdash\; f : B$$

into the term

$$\Gamma \;\vdash\; \lambda x . f \;:\; A \multimap B$$

does not alter the associated strategy, simply reorganizes it.

Cartesian product

Given two simple games A and B, define

$$A \quad \& \quad B$$

as the simple game

$$M_{A\&B} = M_A + M_B$$

$$\lambda_{A\&B} = \lambda_A + \lambda_B$$

$$P_{A\&B} = P_A \oplus P_B$$

where $P_A \oplus P_B$ is the **coalesced sum** of the pointed sets P_A and P_B.

This means that every **nonempty** play in $A\&B$ is either in A or in B.

The cartesian product

For every simple game X, there exists an isomorphism

$$\boxed{X \multimap A \& B \quad \cong \quad (X \multimap A) \;\&\; (X \multimap B)}$$

This means that the game $A\&B$ is the **cartesian product** of A and B.

The cartesian product

For every game X, there exists a bijection between the strategies

$$X \longrightarrow A \& B$$

and the pair of strategies

$$X \longrightarrow A \qquad\qquad X \longrightarrow B.$$

The cartesian product

Guess the two strategies

$$\pi_1 \ : \ A\&B \multimap A \qquad\qquad \pi_2 \ : \ A\&B \multimap B$$

such that for every pair

$$f \ : \ X \multimap A \qquad\qquad g \ : \ X \multimap B$$

there exists a unique strategy

$$h \ : \ X \multimap A\&B$$

making the diagram commute:

$$\begin{tikzcd}
 & & A \\
X \arrow[urr, bend left, "f"] \arrow[r, dotted, "h"] \arrow[drr, bend right, "g"'] & A\&B \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
 & & B
\end{tikzcd}$$

Categories of games as completions

A categorical reconstruction of simple games

A categorical reconstruction

Definition. A **duality functor** on a category $\mathscr{C}$ is a functor

$$D \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}^{op}$$

equipped with a natural bijection

$$\varphi_{A,B} \quad : \quad \mathscr{C}(A, DB) \quad \cong \quad \mathscr{C}(B, DA).$$

Theorem.

The category *Games* is the free cartesian category $\mathscr{C}$ with a duality functor.

A categorical reconstruction

Observation. Every duality functor D induces an adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[rr, bend left, "D"] \arrow[rr, phantom, "\perp"] & & \mathscr{C}^{op} \arrow[ll, bend left, "D"] \end{tikzcd}$$

witnessed by the series of bijection:

$$\mathscr{C}\,(A, DB) \quad \cong \quad \mathscr{C}\,(B, DA) \quad \cong \quad \mathscr{C}^{\,op}\,(DA, B)$$

A categorical reconstruction

So, the category *Games* coincides with the free adjunction

$$\begin{tikzcd} (\mathscr{A}, \&, \mathrm{true}) \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & (\mathscr{B}, \oplus, \mathrm{false}) \arrow[ll, bend left, "R"] \end{tikzcd}$$

where

- the category $\mathscr{A}$ has finite products noted $\&$ and true,
- the category $\mathscr{B}$ has finite sums noted $\oplus$ and false.

A categorical reconstruction

Accordingly, the category $\Sigma \mathit{Games}$ coincides with the free adjunction

$$\begin{tikzcd}
(\mathscr{A}, \oplus, \mathrm{false}) \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & (\mathscr{B}, \&, \mathrm{true}) \arrow[ll, bend left, "R"]
\end{tikzcd}$$

where

- the category $\mathscr{A}$ has finite sums noted $\oplus$ and false,
- the category $\mathscr{B}$ has finite products noted $\&$ and true.

Here we note $\Sigma\mathscr{C}$ for the free category with finite sums generated by $\mathscr{C}$

In particular...

- The simple game for the booleans is defined as

$$\mathbb{B} \quad = \quad R\,(\,L(\mathrm{true}) \oplus L(\mathrm{true})\,)$$

- The tensor product of two simple games

$$A = R \bigoplus_i LA_i \qquad\qquad B = R \bigoplus_j LB_j$$

is defined as

$$A \otimes B \quad = \quad A \oslash B \quad \& \quad A \varodot B$$

where

$$A \oslash B = R \bigoplus_i L(A_i \otimes B) \qquad\qquad A \varodot B = R \bigoplus_j L(A \otimes B_j)$$

Work in progress

An adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathscr{B} \arrow[ll, bend left, "R"] \end{tikzcd}$$

is called bicomplete when

- $\mathscr{A}$ has small colimits
- $\mathscr{B}$ has small limits

Ongoing work:

Describe the bicompletion extending Whitman's construction to categories.