Sheaves and Duality

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Introduction

- A sheaf representation of an abstract algebra is a topological decomposition of the algebra into simpler 'stalks'.
- A distributive lattice of commuting congruences has long been known to be an essential ingredient for a 'good' sheaf representation.
- Our aims here:
  - characterize these 'good' sheaf representations,
  - dualize these sheaf representations using our characterization.
- These results unify and generalize existing results on sheaf representations and duality for Boolean products, MV-algebras, Gelfand rings, and other algebras.

Definition of étale space

- Let $\mathcal{V}$ be a variety of abstract algebras.
- Let (Y, ρ) be a topological space.
- Let $(A_y)_{y \in Y}$ be a Y-indexed family of $\mathcal{V}$-algebras.
- Let $E := \coprod_{y \in Y} A_y$, with $p : E \twoheadrightarrow Y$ the natural surjection.
- Suppose τ is a topology on E such that
  - $p : (E, \tau) \twoheadrightarrow (Y, \rho)$ is a local homeomorphism: any point has an open neighbourhood on which p has a right inverse.
  - Every operation of A_y is continuous in $\tau|_{A_y}$.
- Then $p : (E, \tau) \twoheadrightarrow (Y, \rho)$ is called an étale space of $\mathcal{V}$-algebras.

Sheaf from an étale space

- Let $p : (E, \tau) \twoheadrightarrow (Y, \rho)$ be an étale space of $\mathcal{V}$-algebras.
- For any $U \in \rho$, write FU for the set of local sections over U:

$$FU := \{s : U \to E \text{ continuous s.t. } p \circ s = \mathbf{id}_U\}.$$

- Note: FU is a subalgebra of $\prod_{y \in U} A_y$, and hence in $\mathcal{V}$.
- If $U \subseteq V$, there is a natural restriction map $FV \to FU$.
- F is called the sheaf associated to the étale space.

Definition of sheaf

- A sheaf F on Y consists of the data:
  - For each open U, a $\mathcal{V}$-algebra FU ("local sections");
  - For each open $U \subseteq V$, a $\mathcal{V}$-homomorphism $-|_U : FV \to FU$ ("restriction maps");

  such that F is functorial and has the patching property:
  - For any open cover $(U_i)_{i \in I}$ of an open set U, and any "compatible family" of local sections $(s_i)_{i \in I}$, i.e., $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$,
  - there exists a unique $s \in FU$ such that $s|_{U_i} = s_i$ for all $i \in I$.
- FY is called the algebra of global sections of the sheaf F.
- If A is an algebra isomorphic to FY, then F is called a sheaf representation of A.

Sheaves and étale spaces

Fact

The assignment which sends an étale space to its sheaf of local sections is a bijection between étale spaces and sheaves.

Note: although a sheaf F is initially only defined on the open sets of Y, we may use the associated étale space of F to define, for an arbitrary subset S of Y, FS to be the set of local sections over S.

Sheaves and congruences

- Let F be a sheaf representation of A over a space Y with associated étale space $p\colon E \to Y$.
- For each subset S of Y, we have a congruence on A,

$$\theta_F(S) := \ker(-|_S) = \{(a, b) \in A^2 \mid s_a|_S = s_b|_S\}.$$

- In general, there is no reason for $A \to FS$ to be surjective; so $A/\theta_F(S)$ may be a subalgebra of FS.
- But if it is surjective often enough, then a collection of congruences suffices to describe the sheaf.

Stably compact spaces

- Many interesting sheaf representations use a base space which is spectral or compact Hausdorff.
- Stably compact spaces form a common generalization of these two classes.

Stably compact spaces

- "Generalisation of compact Hausdorff to T_0-setting"

Definition

Stably compact space =

- T_0,
- Sober,
- Locally compact,
- Intersection of compact saturated is compact.

A map between stably compact spaces is proper if it is continuous, and the inverse image of any compact saturated set is compact.

Co-compact dual and patch topology

- For any stably compact space (Y, ρ), the collection of compact saturated sets, $\mathcal{K}Y$, is closed under finite unions and arbitrary intersections. The co-compact dual of ρ, ρ^∂, is the topology of complements of compact saturated sets.
- Fact: If (Y, ρ) is stably compact, then so is $Y^\partial := (Y, \rho^\partial)$.
- Define $\rho^p := \rho \vee \rho^\partial$, the patch topology.
- Fact: (Y, ρ^p) is a compact Hausdorff space.
- Let $y \leq y' \iff y' \in \overline{\{y\}}$, the specialization order of ρ.
- Fact: $\leq$ is a closed subspace of $(Y \times Y, \rho^p \times \rho^p)$.

Compact ordered spaces

- A compact ordered space is a tuple $(Y, \pi, \leq)$ where (Y, π) is compact and $\leq$ is a partial order on Y which is a closed subset of the product $Y \times Y$ (Nachbin 1965).
- So $(Y, \rho^p, \leq)$ is a compact ordered space whenever (Y, ρ) is stably compact.
- Given a compact ordered space $(Y, \pi, \leq)$, denote by $\pi^{\downarrow}$ the topology of open down-sets.
- Then $(Y, \pi^{\downarrow})$ is a stably compact space, and $(\pi^{\downarrow})^{\partial} = \pi^{\uparrow}$.

Fact

The categories of stably compact spaces and compact ordered spaces are isomorphic.

Stone and Priestley duality

- Let A be a bounded distributive lattice.
- Let X be the set of prime filters of A.
- For any $a \in A$, define $\widehat{a} := \{x \in X \mid a \in x\}$.
- The Stone topology σ on X is generated by the sets of the form $\widehat{a}$, for $a \in A$.
- The Priestley topology π on X is generated by the sets of the form $\widehat{a} \cap (\widehat{b})^c$, for $a, b \in A$, and the Priestley order $\leq$ is reverse order inclusion.
- The sets of the form $\widehat{a}$ are exactly the compact-opens of (X, σ) and the clopen down-sets of $(X, \pi, \leq)$.

Stone vs. Priestley duality

- DL: category of bounded distributive lattices.
- Stone (1937): DL is dually equivalent to Stone spaces, i.e., sober T_0 spaces whose compact-open sets form a lattice basis for the topology.
- Priestley (1970): DL is dually equivalent to Priestley spaces, i.e., totally order-disconnected compact ordered spaces.

Fact

Spectral spaces form a full subcategory of stably compact spaces, which corresponds to the category of Priestley spaces under the isomorphism between stably compact spaces and compact ordered spaces.

Soft sheaves

Definition

A sheaf F over a space Y is called soft if any local section over a compact saturated subset K of Y can be extended to a global section.

Here, a subset is saturated if it is an intersection of open sets.

Sheaves on stably compact spaces

- Let F be a soft sheaf representation of an algebra A over a stably compact space $Y^{\uparrow}$.
- For every compact saturated set K of $Y^{\uparrow}$, we have the congruence $\theta_F(K)$, and FK is isomorphic to $A/\theta_F(K)$.

Proposition

The function $\theta_F \colon (\mathcal{K}Y^{\uparrow})^{\mathrm{op}} \to \mathrm{Con}\, A$ is a frame homomorphism for which any two congruences in the image commute.

From a frame homomorphism to a sheaf

- Let $\theta\colon (\mathcal{K}Y^{\uparrow})^{\mathrm{op}} \to \mathrm{Con}\, A$ be a frame homomorphism for which any two congruences in the image commute.
- For any $y \in Y$, $\uparrow y$ is compact-saturated, so we may define a stalk A_y by $A/\theta(\uparrow y)$.
- With an appropriate topology, $E_\theta := \bigsqcup_{y \in Y} A_y$ is an étale space over $Y^{\uparrow}$.
- We denote by F_θ the associated sheaf of local sections.

Theorem (Characterization of soft sheaves)

The assignments $F \mapsto \theta_F$ and $\theta \mapsto F_\theta$ are mutually inverse, up to sheaf isomorphism.

Duality yoga

- The Theorem shows that soft sheaf representations of A over $Y^{\uparrow}$ correspond to frame homomorphisms $(\mathcal{K}Y^{\uparrow})^{\mathrm{op}} \to \operatorname{Con} A$ for which any two congruences in the image commute.
- By definition, the open set frame, $\Omega Y^{\downarrow}$, of $Y^{\downarrow}$, consists of the complements of the sets in $\mathcal{K}Y^{\uparrow}$.
- Thus, $\Omega Y^{\downarrow}$ and $(\mathcal{K}Y^{\uparrow})^{\mathrm{op}}$ are isomorphic.
- Soft sheaf representations therefore also correspond to frame homomorphisms $\Omega Y^{\downarrow} \to \operatorname{Con} A$ for which any two congruences in the image commute.

The case of distributive lattices

- Let A be a distributive lattice with Priestley space $(X, \pi, \leq)$.
- Priestley duality implies that the lattice of congruences on A is isomorphic to the frame of open subsets of (X, π).
- Indeed, an isomorphism $\psi_A \colon \operatorname{Con} A \to \Omega X$ is defined by

$$\psi_A(\theta) := \bigcup_{(a,b)\in\theta} (\widehat{a} \cap (\widehat{b})^c).$$

- Thus, frame homomorphisms $\Omega Y^{\downarrow} \to \operatorname{Con} A$ may be viewed as frame homomorphisms $\Omega Y^{\downarrow} \to \Omega X$.
- The latter correspond to continuous functions $X \to Y^{\downarrow}$, by pointfree duality.

Pointfree duality

Proposition (Papert & Strauss)

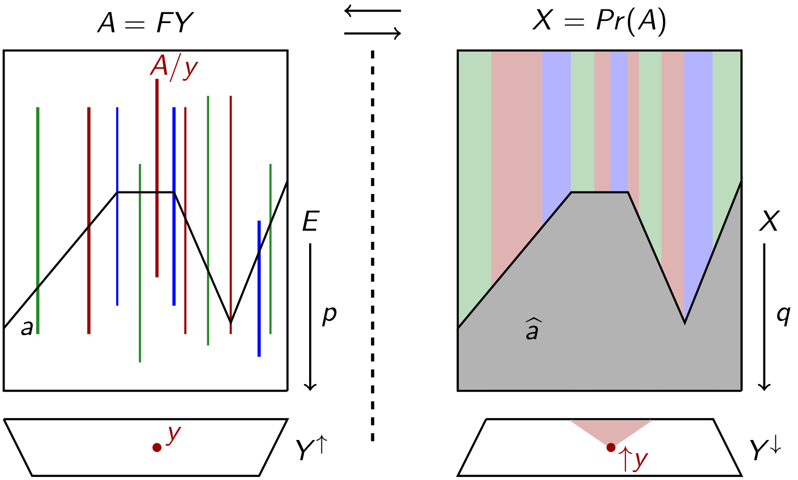
Let X and Y be sober T_0 spaces. For every frame homomorphism $h\colon \Omega Y \to \Omega X$, there is a unique continuous function $f : X \to Y$ such that $h = f^{-1}$.

This gives a dual equivalence between the category of sober T_0 spaces and the category of spatial frames.

The dual of a sheaf

- Let A be a distributive lattice with dual Priestley space X.
- Let F be a soft sheaf representation of A over $Y^{\uparrow}$.
- Then θ_F is a frame homomorphism $\Omega Y^{\downarrow} \to \operatorname{Con} A$ such that any two congruences in the image commute.
- Then $\psi_A \theta_F \colon \Omega Y^{\downarrow} \to \Omega X$ is a frame homomorphism.
- Define q_F to be the continuous function $X \to Y^{\downarrow}$ dual to $\psi_A \theta_F$.

The dual of a sheaf



Duality for commuting congruences

- To a soft sheaf representation F of A over $Y^{\uparrow}$, we have associated a continuous function $q_F : X \to Y^{\downarrow}$.
- How is the commutativity of congruences in the image of θ_F reflected in q_F?

Proposition

Let θ_1, θ_2 be congruences on A and C_1, C_2 the closed sets corresponding to them, i.e., $C_i := X \setminus \psi_A(\theta_i)$. The following are equivalent:

1. *The congruences θ_1 and θ_2 commute.*
2. *For any $x_1 \in C_1$, $x_2 \in C_2$, if $\{i, j\} = \{1, 2\}$ and $x_i \leq x_j$, then there exists $z \in C_1 \cap C_2$ such that $x_i \leq z \leq x_j$.*

Interpolating decompositions

Definition

Let X be a Priestley space and $Y^{\downarrow}$ a stably compact space. A continuous function $q\colon X \to Y^{\downarrow}$ is called an interpolating decomposition of X over Y if, for any $x_1, x_2 \in X$, whenever $x_1 \leq x_2$, there exists $z \in X$ such that $x_1 \leq z \leq x_2$ and $q(z) \geq q(x_1), q(x_2)$.

Sheaves and duality

Theorem (Duality for soft sheaf representations)

Let A be a distributive lattice with Priestley space X, and Y a compact ordered space.

Soft sheaf representations of A over $Y^{\uparrow}$ correspond one-to-one to interpolating decompositions of X over $Y^{\downarrow}$.

Boolean products

Our results generalize the following prototypes:

Theorem (Comer 1971, Burris & Werner 1980)

Boolean product representations of an algebra A are in a natural one-to-one correspondence with relatively complemented distributive lattices of permuting congruences on A.

Theorem (Gehrke 1991)

Boolean product representations of a distributive lattice $A \rightarrowtail \prod_{y \in Y} A_y$ are in a natural one-to-one correspondence with Boolean sum decompositions of the Stone dual space X of A into the Stone dual spaces $(X_y)_{y \in Y}$ of the lattices $(A_y)_{y \in Y}$.

MV-algebras

- MV-algebras are the unit intervals in certain abelian groups with a distributive lattice order.
- The dual Priestley space of (the DL reduct of) an MV-algebra admits at least two distinct interpolating decompositions.
- Our result explains in a simple manner why MV-algebras admit these two soft sheaf representations and how they are related.

Principal congruences of an MV-algebra

A simple but important fact in the representation theory of MV-algebras is that

$$\begin{array}{rcl} \theta : A & \longrightarrow & Con(A) \\ a & \longmapsto & \theta(a) = <(0,a)>_{Con(A)} \end{array}$$

is a bounded lattice homomorphism.

The image of this map is the lattice $Con_{fin}(A)$ of finitely generated MV-algebra congruences of A. Thus, these congruences are pairwise permuting.

The MV-spectrum of A, is the dual space, Y, of $Con_{fin}(A)$

The MV-spectrum as a subspace of the dual space

Since $A \twoheadrightarrow Con_{fin}(A)$ is a bounded distributive lattice quotient, by duality, $Y \hookrightarrow X$ may be seen as a closed subspace of X:

$$Y = \{y \in X \mid I_y \text{ is closed under } \oplus\}$$

We will mainly consider Y in its spectral topology and its dual spectral topology. These are equal to the subspace topologies for the spectral and dual spectral topologies on X, respectively.

The MV-spectrum directly from the MV-algebra

The congruences of an MV-algebra are in 1-to-1 correspondence with MV-ideals: non-empty downsets closed under $\oplus$.

The MV-spectrum may also be seen as the set of those MV-ideals that are prime in the sense that one of $a \ominus b (:= \neg(\neg a \oplus b))$ and $b \ominus a$ is a member for all $a, b \in A$. This is the same set $Y \subseteq X$.

The spectral topology on Y as determined on the previous slide is also the hull-kernel or spectral topology corresponding to the MV-ideals of A.

The maximal MV-spectrum

Given an MV-algebra, A, the subspace Z of Y of maximal MV-ideals of A is called the maximal MV-spectrum. It is compact Hausdorff, but not in general spectral.

Examples

- If $A =$ the free n-generated MV-algebra, then Z is homeomorphic to the cube $[0,1]^n$ with the Euclidean topology.
  - Free_n embeds in $C([0,1]^n, [0,1])$ but the embedding is not unique.
- If A is a Boolean algebra, then Z is its Stone dual space.
- If A is any chain, then Z is the one-point space.
- If A has infinitesimals, then we do not have $A \hookrightarrow C(Z, [0,1])$.

Well-known facts from the literature:

The following are equivalent:

- A bounded distributive lattice D is normal: $a \vee b = 1 \implies \exists c, d \in A$ with $c \wedge d = 0$ and $a \vee d = 1$ and $c \vee b = 1$.
- Each point in $Pr(D)$ is below a unique maximal point.
- The inclusion of the maximal points of the dual space of D admits a continuous retraction

For any MV-algebra A, the lattice $Con_{fin}(A)$ is relatively normal (that is, each interval $[a, b]$ is a normal lattice).

Thus Y is a root-system, that is, $\uparrow y$ is a chain for each $y \in Y$, Z is compact Hausdorff, and the map

$$m : Y \longrightarrow Z, \ y \mapsto \text{ unique maximal point above } y$$

is a continuous retraction

The map k

There is a continuous retraction $k : (X, \sigma^p) \longrightarrow (Y, \sigma^\downarrow)$ (already present in the work of Martínez)

This map may be given a simple description:

$$k(x) = \max\{z \in X \mid I_x \oplus I_z \subseteq I_x\}$$

yielding

(Interpolation Lemma) If $x \leq x'$ then there is x'' with

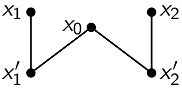
$$x \leq x'' \leq x' \quad \text{and} \quad k(x'') \geq k(x) \quad \text{and} \quad k(x'') \geq k(x')$$

From X to Z without using the MV structure

Combining the two earlier retractions we get

$$m \circ k : (X, \sigma^p) \longrightarrow (Z, \sigma^{\downarrow})$$

The kernel of this map is given by the relation $x_1 W x_2$ iff there are $x_1', x_2', x_0 \in X$ with



Kaplansky's theorem

[Kaplansky 1947]

Let Z_1, Z_2 be compact Hausdorff spaces such that the lattices $C(Z_1, [0,1])$ and $C(Z_2, [0,1])$ are isomorphic. Then Z_1 and Z_2 are homeomorphic spaces.

Kaplansky theorem for arbitrary MV-algebras

Theorem

If A_1 and A_2 are MV-algebras having isomorphic lattice reducts, then the max MV-spectra of A_1 and A_2 are homeomorphic.

- Note that the max MV-spectrum of an MV-algebra of the form $C(Z, [0, 1])$ is Z so that our result generalizes Kaplansky's result.

Proof (sketch).

The maximal MV-spectrum can be reconstructed from the lattice spectrum using the relation W. $\square$

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