Mathematical Aspects of Many-Valued Logics

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My plan

1. formally present two prominent fuzzy logics:
   - Gödel–Dummett logic
   - Łukasiewicz logic

   and to do so in two different ways:
   - using (almost classical) Hilbert style axiomatization
   - using (highly non-classical, yet surprisingly 'natural') semantics
2. give a (hint of a) proof of the completeness theorem (the equality of these two ways)
   - for the Gödel–Dummett logic and
   - show why the same proof would not work for Łukasiewicz logic and how to overcome this problem
3. show that the same proof would work in a much more general setting for an arbitrary logic satisfying certain minimal conditions

The basic syntax: no change there

We consider primitive connectives $\mathcal{L} = \{\overline{0}, \wedge, \vee, \to\}$ and defined connectives $\neg$, $\overline{1}$, and $\leftrightarrow$:

$$\neg\varphi = \varphi \to \overline{0} \qquad \overline{1} = \neg\overline{0} \qquad \varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$$

Formulae are built from a fixed countable set of atoms using the connectives.

Let us by $Fm_{\mathcal{L}}$ denote the set of all formulae.

Recall the semantics of classical logic

Definition 2.1

A **2**-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to $\{0, 1\}$; s.t.:

- $e(\overline{0}) = \overline{0}^{\mathbf{2}} = 0$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{\mathbf{2}} e(\psi) = \min\{e(\varphi), e(\psi)\}$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{\mathbf{2}} e(\psi) = \max\{e(\varphi), e(\psi)\}$
- $e(\varphi \to \psi) = e(\varphi) \to^{\mathbf{2}} e(\psi) = \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi), \\ 0 & \text{otherwise.} \end{cases}$

Definition 2.2

A formula φ is a logical consequence of a theory T (in classical logic), $T \models_{\mathbf{2}} \varphi$, if for every **2**-evaluation e:

$$\text{if } e(\gamma) = 1 \text{ for every } \gamma \in T, \text{ then } e(\varphi) = 1.$$

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- $e(\varphi \to \psi) = e(\varphi) \to^{\mathbf{2}} e(\psi) = \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi), \\ e(\psi) & \text{otherwise.} \end{cases}$

Definition 2.2

A formula φ is a logical consequence of a theory T (in classical logic), $T \models_{\mathbf{2}} \varphi$, if for every **2**-evaluation e:

if $e(\gamma) = 1$ for every $\gamma \in T$, then $e(\varphi) = 1$.

Changing the semantics

Definition 2.3

A $[0,1]_{\mathrm{G}}$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to $[0,1]$; s.t.:

- $e(\overline{0}) = \overline{0}^{[0,1]_{\mathrm{G}}} = 0$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{[0,1]_{\mathrm{G}}} e(\psi) = \min\{e(\varphi), e(\psi)\}$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{[0,1]_{\mathrm{G}}} e(\psi) = \max\{e(\varphi), e(\psi)\}$
- $e(\varphi \to \psi) = e(\varphi) \to^{[0,1]_{\mathrm{G}}} e(\psi) = \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi), \\ e(\psi) & \text{otherwise.} \end{cases}$

Definition 2.4

A formula φ is a logical consequence of a theory T (in Gödel–Dummett logic), $T \models_{[0,1]_{\mathrm{G}}} \varphi$, if for every $[0,1]_{\mathrm{G}}$-evaluation e:

$$\text{if } e(\gamma) = 1 \text{ for every } \gamma \in T, \text{ then } e(\varphi) = 1.$$

Changing the semantics

Some classical properties fail in $\models_{[0,1]_G}$:

- $\not\models_{[0,1]_G} \neg\neg\varphi \to \varphi$ $\qquad \neg\neg\frac{1}{2} \to \frac{1}{2} = 1 \to \frac{1}{2} = \frac{1}{2}$
- $\not\models_{[0,1]_G} \varphi \vee \neg\varphi$ $\qquad \frac{1}{2} \vee \neg\frac{1}{2} = \frac{1}{2}$
- $\not\models_{[0,1]_G} \neg(\neg\varphi \wedge \neg\psi) \to \varphi \vee \psi$ $\qquad \neg(\neg\frac{1}{2} \wedge \neg\frac{1}{2}) \to \frac{1}{2} \vee \frac{1}{2} = 1 \to \frac{1}{2} = \frac{1}{2}$
- $\not\models_{[0,1]_G} ((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$
  $((\frac{1}{2} \to 0) \to 0) \to ((0 \to \frac{1}{2}) \to \frac{1}{2}) = 1 \to \frac{1}{2} = \frac{1}{2}$

Recall a proof system for classical logic

The axioms are:

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \to (\psi \to \varphi)$
(A3)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(A4)	$(\varphi \to \psi) \vee (\psi \to \varphi)$
(A5a)	$\varphi \wedge \psi \to \varphi$
(A5b)	$\varphi \wedge \psi \to \psi$
(A5c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$
(A6a)	$\varphi \to \varphi \vee \psi$
(A6b)	$\psi \to \varphi \vee \psi$
(A6c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
(A7)	$\overline{0} \to \varphi$
(A8)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$
(A9)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

The only inference rule is *modus ponens:* from $\varphi \to \psi$ and φ infer ψ.

We write $T \vdash_{\mathrm{CL}} \varphi$ if there is a proof of φ from T in classical logic.

A proof system for Gödel–Dummett logic

The axioms are:

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \to (\psi \to \varphi)$
(A3)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(A4)	$(\varphi \to \psi) \vee (\psi \to \varphi)$
(A5a)	$\varphi \wedge \psi \to \varphi$
(A5b)	$\varphi \wedge \psi \to \psi$
(A5c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$
(A6a)	$\varphi \to \varphi \vee \psi$
(A6b)	$\psi \to \varphi \vee \psi$
(A6c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
(A7)	$\overline{0} \to \varphi$
(A8)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$

The only inference rule is *modus ponens:* from $\varphi \to \psi$ and φ infer ψ.

We write $T \vdash_{\mathrm{G}} \varphi$ if there is a proof of φ from T in Gödel–Dummett logic.

Completeness theorem for classical logic

Theorem 2.5

For every theory T and a formula φ we have:

$$T \vdash_{\mathrm{CL}} \varphi \text{ if, and only if, } T \models_{\mathbf{2}} \varphi.$$

Completeness theorem for Gödel–Dummett logic

Theorem 2.6

For every theory T and a formula φ we have:

$$T \vdash_{\mathrm{G}} \varphi \textit{ if, and only if, } T \models_{[0,1]_{\mathrm{G}}} \varphi.$$

Recall a proof system for classical logic

The axioms are:

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \to (\psi \to \varphi)$
(A3)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(A4)	$(\varphi \to \psi) \vee (\psi \to \varphi)$
(A5a)	$\varphi \wedge \psi \to \varphi$
(A5b)	$\varphi \wedge \psi \to \psi$
(A5c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$
(A6a)	$\varphi \to \varphi \vee \psi$
(A6b)	$\psi \to \varphi \vee \psi$
(A6c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
(A7)	$\overline{0} \to \varphi$
(A8)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$
(A9)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

The only inference rule is *modus ponens:* from $\varphi \to \psi$ and φ infer ψ.

A proof system for Gödel–Dummett logic

The axioms are:

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \to (\psi \to \varphi)$
(A3)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(A4)	$(\varphi \to \psi) \vee (\psi \to \varphi)$
(A5a)	$\varphi \wedge \psi \to \varphi$
(A5b)	$\varphi \wedge \psi \to \psi$
(A5c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$
(A6a)	$\varphi \to \varphi \vee \psi$
(A6b)	$\psi \to \varphi \vee \psi$
(A6c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
(A7)	$\overline{0} \to \varphi$
(A8)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$

The only inference rule is *modus ponens:* from $\varphi \to \psi$ and φ infer ψ.

Relation to intuitionistic logic

The intuitionistic logic has axioms:

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \to (\psi \to \varphi)$
(A3)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(A5a)	$\varphi \wedge \psi \to \varphi$
(A5b)	$\varphi \wedge \psi \to \psi$
(A5c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$
(A6a)	$\varphi \to \varphi \vee \psi$
(A6b)	$\psi \to \varphi \vee \psi$
(A6c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
(A7)	$\overline{0} \to \varphi$
(A8)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$

The only inference rule is *modus ponens:* from $\varphi \to \psi$ and φ infer ψ.

Algebraic semantics

A *Heyting-algebra* is a structure $\boldsymbol{B} = \langle B, \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}, \rightarrow^{\boldsymbol{B}}, \overline{0}^{\boldsymbol{B}}, \overline{1}^{\boldsymbol{B}} \rangle$ such that:

(1) $\langle B, \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}, \overline{0}^{\boldsymbol{B}}, \overline{1}^{\boldsymbol{B}} \rangle$ is a bounded lattice,

(2) $z \leq x \rightarrow^{\boldsymbol{B}} y$ iff $x \wedge^{\boldsymbol{B}} z \leq y$, (residuation)

where $x \leq y$ is defined as $x \wedge y = x$ or (equivalently) as $x \rightarrow y = \overline{1}$.

We say that $\boldsymbol{B}$ is

- Gödel algebra (or just G-algebra) whenever

$$(x \rightarrow y) \vee (y \rightarrow x) = \overline{1} \qquad \text{(prelinearity)}$$

- linearly ordered (or Heyting chain) if $\leq$ is a total order.

Note that each Heyting chain is G-algebra, so we also call it G-chain.

By $\mathbb{G}$ (or $\mathbb{G}_{\text{lin}}$ resp.) we denote the class of all G-algebras (G-chains resp.)

Standard semantics

Consider algebra $[0,1]_{\mathrm{G}} = \langle [0,1], \wedge^{[0,1]_{\mathrm{G}}}, \vee^{[0,1]_{\mathrm{G}}}, \rightarrow^{[0,1]_{\mathrm{G}}}, 0, 1 \rangle$, where:

$$a \wedge^{[0,1]_{\mathrm{G}}} b = \min\{a, b\}$$

$$a \vee^{[0,1]_{\mathrm{G}}} b = \max\{a, b\}$$

$$a \rightarrow^{[0,1]_{\mathrm{G}}} b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

Exercise 1 (Easy)

Prove that $[0,1]_{\mathrm{G}}$ is the unique G-chain with the lattice reduct $\langle [0,1], \min, \max, 0, 1 \rangle$.

Recall the notion of $[0,1]_{\mathrm{G}}$-evaluation

Definition 2.7

A $[0,1]_{\mathrm{G}}$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to $[0,1]$; s.t.:

- $e(\overline{0}) = \overline{0}^{[0,1]_{\mathrm{G}}} = 0$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{[0,1]_{\mathrm{G}}} e(\psi) = \min\{e(\varphi), e(\psi)\}$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{[0,1]_{\mathrm{G}}} e(\psi) = \max\{e(\varphi), e(\psi)\}$
- $e(\varphi \to \psi) = e(\varphi) \to^{[0,1]_{\mathrm{G}}} e(\psi) = \cdots$

Definition 2.8

A formula φ is a logical consequence of set of a theory T
in Gödel–Dummett logic, $T \models_{[0,1]_{\mathrm{G}}} \varphi$,
if for every $[0,1]_{\mathrm{G}}$-evaluation e:

if $e(\gamma) = 1$ for every $\gamma \in T$, then $e(\varphi) = 1$.

General notion of semantical consequence

Definition 2.9

A $\boldsymbol{B}$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to B such that:

- $e(\overline{0}) = \overline{0}^{\boldsymbol{B}}$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{\boldsymbol{B}} e(\psi)$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{\boldsymbol{B}} e(\psi)$
- $e(\varphi \to \psi) = e(\varphi) \to^{\boldsymbol{B}} e(\psi)$

Definition 2.10

A formula φ is a logical consequence of a theory T
w.r.t. a class $\mathbb{K}$ of G-algebras, $T \models_{\mathbb{K}} \varphi$,
if for every $\boldsymbol{B} \in \mathbb{K}$ and every $\boldsymbol{B}$-evaluation e:

if $e(\gamma) = \overline{1}^{\boldsymbol{B}}$ for every $\gamma \in T$, then $e(\varphi) = \overline{1}^{\boldsymbol{B}}$.

Three completeness theorems

Theorem 2.11

The following are equivalent for every theory T and a formula φ:

1. $T \vdash_{\mathrm{G}} \varphi$
2. $T \models_{\mathbb{G}} \varphi$ — *w.r.t. general semantics*
3. $T \models_{\mathbb{G}_{\mathrm{lin}}} \varphi$ — *w.r.t. linear semantics*
4. $T \models_{[0,1]_{\mathrm{G}}} \varphi$ — *w.r.t. standard semantics*

Exercise 2 (Medium)

Prove the implications from top to bottom.

Some theorems and derivations in G

Proposition 2.12

(T1) $\vdash_{\mathrm{G}} \varphi \to \varphi$

(T2) $\vdash_{\mathrm{G}} \varphi \to (\psi \to \varphi \wedge \psi)$

(D1) $\varphi \leftrightarrow \overline{1} \vdash_{\mathrm{G}} \varphi$ *and* $\varphi \vdash_{\mathrm{G}} \varphi \leftrightarrow \overline{1}$

(D2) $\varphi \to \psi \vdash_{\mathrm{G}} \varphi \wedge \psi \leftrightarrow \varphi$ *and* $\varphi \wedge \psi \leftrightarrow \varphi \vdash_{\mathrm{G}} \varphi \to \psi$

(D3) $\varphi \to (\psi \to \chi) \vdash_{\mathrm{G}} \varphi \wedge \psi \to \chi$ *and* $\varphi \wedge \psi \to \chi \vdash_{\mathrm{G}} \varphi \to (\psi \to \chi)$

Proposition 2.13

$\vdash_{\mathrm{G}} \varphi \wedge \psi \leftrightarrow \psi \wedge \varphi$

$\vdash_{\mathrm{G}} \varphi \wedge (\psi \wedge \chi) \leftrightarrow (\varphi \wedge \psi) \wedge \chi$

$\vdash_{\mathrm{G}} \varphi \wedge (\varphi \vee \psi) \leftrightarrow \varphi$

$\vdash_{\mathrm{G}} \overline{1} \wedge \varphi \leftrightarrow \varphi$

$\vdash_{\mathrm{G}} (\varphi \to \psi) \vee (\psi \to \varphi) \leftrightarrow \overline{1}$

$\vdash_{\mathrm{G}} \varphi \vee \psi \leftrightarrow \psi \vee \varphi$

$\vdash_{\mathrm{G}} \varphi \vee (\psi \vee \chi) \leftrightarrow (\varphi \vee \psi) \vee \chi$

$\vdash_{\mathrm{G}} \varphi \vee (\varphi \wedge \psi) \leftrightarrow \varphi$

$\vdash_{\mathrm{G}} \overline{0} \vee \varphi \leftrightarrow \varphi$

The rule of substitution

Proposition 2.14

$$\vdash_{\mathrm{G}} \varphi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi \vdash_{\mathrm{G}} \psi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi, \psi \leftrightarrow \chi \vdash_{\mathrm{G}} \varphi \leftrightarrow \chi$$

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{G}} (\varphi \wedge \chi) \leftrightarrow (\psi \wedge \chi) \qquad \varphi \leftrightarrow \psi \vdash_{\mathrm{G}} (\varphi \vee \chi) \leftrightarrow (\psi \vee \chi)$$

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{G}} (\chi \wedge \varphi) \leftrightarrow (\chi \wedge \varphi) \qquad \varphi \leftrightarrow \psi \vdash_{\mathrm{G}} (\chi \vee \varphi) \leftrightarrow (\chi \vee \psi)$$

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{G}} (\varphi \to \chi) \leftrightarrow (\psi \to \chi) \qquad \varphi \leftrightarrow \psi \vdash_{\mathrm{G}} (\chi \to \varphi) \leftrightarrow (\chi \to \psi)$$

Corollary 2.15

$\varphi \leftrightarrow \psi \vdash_{\mathrm{G}} \chi \leftrightarrow \chi'$, *where χ' results from χ by replacing its subformula φ by ψ.*

Exercise 3 (Difficult and tedious; but can be automatized)

Prove this corollary and the three previous propositions.

Lindenbaum–Tarski algebra

Definition 2.16

Let T be a theory. We define

$$[\varphi]_T = \{\psi \mid T \vdash_{\mathrm{G}} \varphi \leftrightarrow \psi\} \qquad L_T = \{[\varphi]_T \mid \varphi \in Fm_{\mathcal{L}}\}$$

The Lindenbaum–Tarski algebra of a theory T ($\mathbf{LindT}_T$) as an algebra with the domain L_T and operations:

$$\overline{0}^{\mathbf{LindT}_T} = [\overline{0}]_T$$
$$[\varphi]_T \to^{\mathbf{LindT}_T} [\psi]_T = [\varphi \to \psi]_T$$
$$[\varphi]_T \vee^{\mathbf{LindT}_T} [\psi]_T = [\varphi \vee \psi]_T$$
$$[\varphi]_T \wedge^{\mathbf{LindT}_T} [\psi]_T = [\varphi \wedge \psi]_T$$

Exercise 4 (Easy)

Prove that the definition of $\mathbf{LindT}_T$ is sound.

Lindenbaum–Tarski algebra: basic properties

Proposition 2.17

1. $[\varphi]_T = \overline{1}^{\mathbf{LindT}_T}$ *iff* $T \vdash_{\mathrm{G}} \varphi$.
2. $[\varphi]_T \leq [\psi]_T$ *iff* $T \vdash_{\mathrm{G}} \varphi \to \psi$.
3. $\mathbf{LindT}_T$ *is a* G*-algebra.*

Proof.

1. $[\varphi]_T = [\overline{1}]_T$ iff $T \vdash_{\mathrm{G}} \varphi \leftrightarrow \overline{1}$ iff (by (D1)) $T \vdash_{\mathrm{G}} \varphi$.

2. $[\varphi]_T \leq [\psi]_T$ iff $[\varphi]_T \wedge [\psi]_T = [\varphi]_T$ iff $[\varphi \wedge \psi]_T = [\varphi]_T$ iff $T \vdash_{\mathrm{G}} \varphi \wedge \psi \leftrightarrow \varphi$ iff (by (D2)) $T \vdash_{\mathrm{G}} \varphi \to \psi$.

3. The validity of identities follows from Proposition 2.13 and the residuation from (D3): $[\varphi]_T \leq [\psi]_T \to [\chi]_T$ iff $T \vdash \varphi \to (\psi \to \chi)$ iff $T \vdash_{\mathrm{G}} \varphi \wedge \psi \to \chi$ iff $[\varphi]_T \wedge [\psi]_T \leq [\chi]_T$.

□

Lindenbaum–Tarski algebras

Proposition 2.18

For each theory T there is a G-algebra $\mathbf{LindT}_T$ (called the Lindenbaum–Tarski algebra of the theory T) and an $\mathbf{LindT}_T$-evaluation e_T st

1. $e_T(\chi) = \overline{1}^{\mathbf{LindT}_T}$ *iff* $T \vdash \chi$.
2. $\mathbf{LindT}_T$ *is a* G*-chain iff* $T \vdash_{\mathrm{G}} \psi \to \chi$ *or* $T \vdash_{\mathrm{G}} \chi \to \psi$ *for each* ψ, χ.

A theory T is linear if $T \vdash_{\mathrm{G}} \varphi \to \psi$ or $T \vdash_{\mathrm{G}} \psi \to \varphi$ for each φ, ψ.

Exercise 5 (Easy)

Prove it.

The proof of the first (general) completeness

Theorem 2.19

For every theory T and a formula φ we have:

$$T \vdash_{\mathrm{G}} \varphi \text{ if, and only if, } T \models_{\mathbb{G}} \varphi.$$

Proof.

Assume that $T \nvdash_{\mathrm{G}} \varphi$

Take the G-algebra $\mathbf{LindT}_T$

Take the $\mathbf{LindT}_T$-evaluation e_T, recall that $e_T(\chi) = \bar{1}^{\mathbf{LindT}_T}$ iff $T \vdash \chi$

Clearly $e_T(\varphi) \neq \bar{1}^{\mathbf{LindT}_T}$ and

$e_T(\chi) = \bar{1}^{\mathbf{LindT}_T}$ for each $\chi \in T$ (because $T \vdash_{\mathrm{G}} \chi$) □

The proof of the second (linear) completeness

Theorem 2.20

For every theory T and a formula φ we have:

$$T \vdash_{\mathrm{G}} \varphi \text{ if, and only if, } T \models_{\mathbb{G}_{\mathrm{lin}}} \varphi.$$

Proof.

Assume that $T \nvdash_{\mathrm{G}} \varphi$ and that there is linear $S \supseteq T$ st. $S \nvdash_{\mathrm{G}} \varphi$

Take the G-algebra $\mathbf{LindT}_S$, we know that it is a G-chain

Take the $\mathbf{LindT}_S$-evaluation e_S, recall that $e_S(\chi) = \overline{1}^{\mathbf{LindT}_{T'}}$ iff $S \vdash \chi$

Clearly $e_S(\varphi) \neq \overline{1}^{\mathbf{LindT}_S}$ and

$e_S(\chi) = \overline{1}^{\mathbf{LindT}_S}$ for each $\chi \in T$ (because $S \vdash_{\mathrm{G}} \chi$) □

Linear Extension Property

A theory T is linear if $T \vdash_{\mathrm{G}} \varphi \to \psi$ or $T \vdash_{\mathrm{G}} \psi \to \varphi$ for each φ, ψ.

Lemma 2.21 (Linear Extension Property)

If $T \nvdash_{\mathrm{G}} \varphi$, then there is linear theory $S \supseteq T$ s.t. $S \nvdash_{\mathrm{G}} \varphi$.

Proof.

Assume that $T \nvdash_{\mathrm{L}} \varphi$

Take a maximal $S \supseteq T$ s.t. $S \nvdash_{\mathrm{L}} \varphi$ (it exists due to the Zorn's lemma)

Assume that S is not linear, i.e., there are formulae ψ, χ s.t. $S \nvdash_{\mathrm{L}} \psi \to \chi$ and $S \nvdash_{\mathrm{L}} \chi \to \psi$

Then, due to the maximality of S: $S, \psi \to \chi \vdash_{\mathrm{L}} \varphi$ and $S, \chi \to \psi \vdash_{\mathrm{L}} \varphi$

If we would know that this implies: $S \vdash_{\mathrm{L}} \varphi$, we have a contradiction! □

Semilinearity Property

Lemma 2.22 (Semilinearity Property)

If $T, \psi \to \chi \vdash_{\mathrm{G}} \varphi$ *and* $T, \chi \to \psi \vdash_{\mathrm{G}} \varphi$*, then* $T \vdash_{\mathrm{G}} \varphi$*.*

Proof.

If we would know that G has the deduction theorem:

$T \vdash_{\mathrm{G}} (\psi \to \chi) \to \varphi$ and $T \vdash_{\mathrm{G}} (\chi \to \psi) \to \varphi$

Thus by axiom (A6c) $T \vdash_{\mathrm{G}} (\psi \to \chi) \vee (\chi \to \psi) \to \varphi$

Then axiom (A4) completes the proof. □

Deduction Theorem

Theorem 2.23 (Deduction theorem)

For every set of formulae $T \cup \{\varphi, \psi\}$,

$$T, \varphi \vdash_{\mathrm{G}} \psi \text{ iff } T \vdash_{\mathrm{G}} \varphi \to \psi$$

Exercise 6 (Medium)

Prove it.

The proof of the third (standard) completeness

Contrapositively: assume that $T \nvdash_{\mathrm{G}} \varphi$. Let $\boldsymbol{B}$ be a countable G-chain[1] and e a $\boldsymbol{B}$-evaluation such that $e[T] \subseteq \{\overline{1}^{\boldsymbol{B}}\}$ and $e(\varphi) \neq \overline{1}^{\boldsymbol{B}}$.

There has to be (because every countable order can be monotonously embedded into a dense one) a mapping $f\colon B \to [0,1]$ such that $f(\overline{0}) = 0, f(\overline{1}) = 1$, and for each $a, b \in B$ we have:

$$a \leq b \quad \text{iff} \quad f(a) \leq f(a)$$

We define a mapping $\bar{e}\colon Fm_{\mathcal{L}} \to [0,1]$ as

$$\bar{e}(\psi) = f(e(\psi))$$

and prove (by induction) that it is $[0,1]_{\mathrm{G}}$-evaluation.

Then $\bar{e}(\psi) = 1$ iff $e(\psi) = \overline{1}^{\boldsymbol{B}}$ and so $\bar{e}[T] \subseteq \{1\}$ and $\bar{e}(\varphi) \neq 1$.

[1] E.g. $\boldsymbol{B} = \mathbf{LindT}_{T'}$ for some linear $T' \supseteq T$ st. $T' \nvdash_{\mathrm{G}} \varphi$.

We still keep the classical syntax

We consider primitive connectives $\mathcal{L} = \{\overline{0}, \wedge, \vee, \to\}$ and defined connectives $\neg$, $\overline{1}$, and $\leftrightarrow$:

$$\neg\varphi = \varphi \to \overline{0} \qquad \overline{1} = \neg\overline{0} \qquad \varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$$

Formulae are built from a fixed countable set of atoms using the connectives.

Let us by $Fm_{\mathcal{L}}$ denote the set of all formulae.

But we also use additional connectives $\oplus$ and $\&$ defined as:

$$\varphi \oplus \psi = \neg\varphi \to \psi \qquad \varphi \,\&\, \psi = \neg(\varphi \to \neg\psi)$$

Recall the semantics of Gödel–Dummett logic

Definition 2.24

A $[0,1]_{\mathrm{G}}$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to $[0,1]$; s.t.:

- $e(\overline{0}) = \overline{0}^{[0,1]_{\mathrm{G}}} = 0$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{[0,1]_{\mathrm{G}}} e(\psi) = \min\{e(\varphi), e(\psi)\}$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{[0,1]_{\mathrm{G}}} e(\psi) = \max\{e(\varphi), e(\psi)\}$
- $e(\varphi \to \psi) = e(\varphi) \to^{[0,1]_{\mathrm{G}}} e(\psi) = \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi), \\ e(\psi) & \text{otherwise.} \end{cases}$

Definition 2.25

A formula φ is a logical consequence of a theory T (in Gödel–Dummett logic), $T \models_{[0,1]_{\mathrm{G}}} \varphi$, if for every $[0,1]_{\mathrm{G}}$-evaluation e:

$$\text{if } e(\gamma) = 1 \text{ for every } \gamma \in T, \text{ then } e(\varphi) = 1.$$

Changing the semantics (again)

Definition 2.26

A $[0,1]_{Ł}$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to $[0,1]$; s.t.:

- $e(\overline{0}) = \overline{0}^{[0,1]_{Ł}} = 0$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{[0,1]_{Ł}} e(\psi) = \min\{e(\varphi), e(\psi)\}$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{[0,1]_{Ł}} e(\psi) = \max\{e(\varphi), e(\psi)\}$
- $e(\varphi \to \psi) = e(\varphi) \to^{[0,1]_{Ł}} e(\psi) = \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi) \\ 1 - e(\varphi) + e(\psi) & \text{otherwise} \end{cases}$

Definition 2.27

A formula φ is a logical consequence of a theory T (in Łukasiewicz logic), $T \models_{[0,1]_{Ł}} \varphi$, if for every $[0,1]_{Ł}$-evaluation e:

$$\text{if } e(\gamma) = 1 \text{ for every } \gamma \in T, \text{ then } e(\varphi) = 1.$$

Changing the semantics (again)

Some classical properties fail in $\models [0,1]_{Ł}$:

- $\not\models_{[0,1]_Ł} \varphi \vee \neg\varphi$ $\qquad \frac{1}{2} \vee \neg\frac{1}{2} = \frac{1}{2}$
- $\not\models_{[0,1]_Ł} (\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$

  $(\frac{1}{2} \to (\frac{1}{2} \to 0)) \to (\frac{1}{2} \to 0) = 1 \to \frac{1}{2} = \frac{1}{2}$

BUT other classical properties hold, e.g.:

- $\models_{[0,1]_Ł} \neg\neg\varphi \to \varphi$
- $\models_{[0,1]_Ł} ((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$
- all De Morgan laws involving $\neg, \vee, \wedge$

Recall a proof system for classical logic

The axioms are:

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \to (\psi \to \varphi)$
(A3)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(A4)	$(\varphi \to \psi) \vee (\psi \to \varphi)$
(A5a)	$\varphi \wedge \psi \to \varphi$
(A5b)	$\varphi \wedge \psi \to \psi$
(A5c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$
(A6a)	$\varphi \to \varphi \vee \psi$
(A6b)	$\psi \to \varphi \vee \psi$
(A6c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
(A7)	$\overline{0} \to \varphi$
(A8)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$
(A9)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

The only inference rule is *modus ponens:* from $\varphi \to \psi$ and φ infer ψ.

We write $T \vdash_{\mathrm{CL}} \varphi$ if there is a proof of φ from T in classical logic.

Recall a proof system for Gödel–Dummett logic

The axioms are:

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \to (\psi \to \varphi)$
(A3)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(A4)	$(\varphi \to \psi) \vee (\psi \to \varphi)$
(A5a)	$\varphi \wedge \psi \to \varphi$
(A5b)	$\varphi \wedge \psi \to \psi$
(A5c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$
(A6a)	$\varphi \to \varphi \vee \psi$
(A6b)	$\psi \to \varphi \vee \psi$
(A6c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
(A7)	$\overline{0} \to \varphi$
(A8)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$

The only inference rule is *modus ponens:* from $\varphi \to \psi$ and φ infer ψ.

We write $T \vdash_{\mathrm{G}} \varphi$ if there is a proof of φ from T in Gödel–Dummett logic.

A proof system for Łukasiewicz logic

The axioms are:

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \to (\psi \to \varphi)$
(A3)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(A4)	$(\varphi \to \psi) \vee (\psi \to \varphi)$
(A5a)	$\varphi \wedge \psi \to \varphi$
(A5b)	$\varphi \wedge \psi \to \psi$
(A5c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$
(A6a)	$\varphi \to \varphi \vee \psi$
(A6b)	$\psi \to \varphi \vee \psi$
(A6c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
(A7)	$\overline{0} \to \varphi$
(A9)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

The only inference rule is *modus ponens:* from $\varphi \to \psi$ and φ infer ψ.

We write $T \vdash_{Ł} \varphi$ if there is a proof of φ from T in Łukasiewicz logic.

Completeness theorem for Łukasiewicz logic

Theorem 2.28

For every finite theory T and a formula φ we have:

$$T \vdash_{Ł} \varphi \text{ if, and only if, } T \models_{[0,1]_Ł} \varphi.$$

Exercise 7 (Easy)

Prove the implication from left to right.

Finitarity vs. compactness of $\models_{[0,1]_Ł}$ and $\vdash_Ł$

Proposition 2.29

1. $\models_{[0,1]_Ł}$ *is compact i.e., if each finite $T' \subseteq T$ there is an $[0,1]_Ł$- evaluation e st. $e[T'] \subseteq \{1\}$, then there is an $[0,1]_Ł$-evaluation e st. $e[T] \subseteq \{1\}$*
2. $\models_{[0,1]_Ł}$ *is not finitary i.e., there is $T \cup \{\varphi\}$ s.t. $T \models_{[0,1]_Ł} \varphi$ but for no finite $T' \subseteq T$ we have $T' \models_{[0,1]_Ł} \varphi$*
3. $\vdash_Ł$ *is finitary*

- $a \oplus b = \min\{a+b, 1\}$ $\qquad \varphi \oplus \psi := \neg\varphi \to \psi$
- $\Sigma = \{(p \oplus \overset{n}{\ldots} \oplus p) \to q \mid n \geq 1\} \cup \{\neg p \to q\}$
- $\Sigma \models_{[0,1]_Ł} q$
- For every finite $\Sigma_0 \subseteq \Sigma$, $\Sigma_0 \not\models_{[0,1]_Ł} q$.

Thus we cannot have the strong completeness theorem $\vdash_Ł = \models_{[0,1]_Ł}$

A problem for the completeness proof

The 'normal' deduction theorem fails in Ł:

Proof.

Clearly $\varphi, \varphi \to (\varphi \to \psi) \vdash_{Ł} \psi$ (by using *modus ponens* twice)

But then by DT twice also $\vdash_{Ł} (\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$

And so by soundness also: $\models_{[0,1]_{Ł}} (\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$ …

…which we know is not true. □

We can only prove a local deduction theorem:

Theorem 2.30

For every theory T and formulae φ and ψ we have:

$$T, \varphi \vdash_{Ł} \psi \text{ iff there is } n \geq 1 \text{ such that } T \vdash_{Ł} \varphi \,\&\, \overset{n}{\ldots} \,\&\, \varphi \to \psi$$

How do we get the Semilinearity Property?

Assume that we would be able to prove:

Theorem 2.31 (Proof by Cases Property)

If $T, \psi \vdash_{Ł} \varphi$ *and* $T, \chi \vdash_{Ł} \varphi$, *then* $T, \psi \vee \chi \vdash_{Ł} \varphi$.

Then the Semilinearity Property easily follows using axiom $(A4)$

$$(\psi \to \chi) \vee (\chi \to \psi) .$$

Lemma 2.32 (Semilinearity Property)

If $T, \psi \to \chi \vdash_{Ł} \varphi$ *and* $T, \chi \to \psi \vdash_{Ł} \varphi$, *then* $T \vdash_{Ł} \varphi$.

A proof of the Proof by Cases Property

Exercise 8 (Medium)

Prove that
(P1) $\vdash_{Ł} \varphi \vee \varphi \to \varphi$
(P2) $\vdash_{Ł} \varphi \vee \psi \to \psi \vee \varphi$
(P3) $\varphi \vee \chi, (\varphi \to \psi) \vee \chi \vdash_{Ł} \psi \vee \chi$

Assume: $T, \psi \vdash_{Ł} \varphi$ and $T, \chi \vdash_{Ł} \varphi$

Assume that we know that: if $T \vdash_{Ł} \varphi$, then $\{\psi \vee \chi \mid \psi \in T\} \vdash_{Ł} \varphi \vee \chi$

Then: $T \vee \chi, \psi \vee \chi \vdash_{Ł} \varphi \vee \chi$ and $T \vee \varphi, \chi \vee \varphi \vdash_{Ł} \varphi \vee \varphi$.

Using (A6a), (P1), and (P2) we get $T, \psi \vee \chi \vdash_{Ł} \chi \vee \varphi$ and $T, \chi \vee \varphi \vdash_{Ł} \varphi$

Thus obviously: $T, \psi \vee \chi \vdash_{Ł} \varphi$

A proof of the Proof by Cases Property

Exercise 8 (Medium)

Prove that

(P1) $\vdash_{Ł} \varphi \vee \varphi \to \varphi$

(P2) $\vdash_{Ł} \varphi \vee \psi \to \psi \vee \varphi$

(P3) $\varphi \vee \chi, (\varphi \to \psi) \vee \chi \vdash_{Ł} \psi \vee \chi$

So we need to show that: if $T \vdash_{Ł} \varphi$, then $\{\psi \vee \chi \mid \psi \in T\} \vdash_{Ł} \varphi \vee \chi$

We prove more: If $T \vdash_{Ł} \varphi$, then $T \vee \chi \vdash_{Ł} \delta \vee \chi$
for each δ appearing in the proof of φ from T.

It is trivial for $\delta \in T$ or δ an axiom

if we used MP, by IH there has to be η st.

$T \vee \chi \vdash_{Ł} \eta \vee \chi$ $\quad$ $T \vee \chi \vdash_{Ł} (\eta \to \delta) \vee \chi$ $\quad$ thus (P3) completes the proof.

Algebraic semantics

An MV-*algebra* is a structure $\boldsymbol{B} = \langle B, \oplus, \neg, \overline{0}\rangle$ such that:

(1) $\langle B, \oplus, \overline{0}\rangle$ is a commutative monoid,

(2) $\neg\neg x = x$,

(3) $x \oplus \neg\overline{0} = \neg\overline{0}$,

(4) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

In each MV-algebra we define additional operations:

$x \to y$	is	$\neg x \oplus y$	implication
$x \mathbin{\&} y$	is	$\neg(\neg x \oplus \neg y)$	strong conjunction
$x \wedge y$	is	$x \mathbin{\&} (x \to y)$	min-conjunction
$x \vee y$	is	$\neg(\neg x \wedge \neg y)$	max-disjunction
$\overline{1}$	is	$\neg\overline{0}$	top

Exercise 9 (Easy)

Prove that $\langle B, \wedge, \vee, \overline{0}, \overline{1}\rangle$ is a bounded lattice.

Algebraic semantics cont. and standard semantics

We say that an MV-algebra $\boldsymbol{B}$ is linearly ordered (or MV-chain) if its lattice reduct is.

By $\mathbb{MV}$ (or $\mathbb{MV}_{\mathrm{lin}}$ resp.) we denote the class of all MV-algebras
(MV-chains resp.)

Take the algebra $[0,1]_{Ł} = \langle [0,1], \oplus, \neg, 0\rangle$, with operations defined as:

$$\neg a = 1 - a \qquad\qquad a \oplus b = \min\{1, a+b\}.$$

Proposition 2.33

$[0,1]_{Ł}$ *is the unique (up to isomorphism)* MV*-chain with the lattice reduct* $\langle [0,1], \min, \max, 0, 1\rangle$.

Exercise 10 (Easy)

Check that $[0,1]_{Ł}$ is an MV-chain and find another MV-chain isomorphic to $[0,1]_{Ł}$ with the same lattice reduct.

General notion of semantical consequence

Definition 2.34

A $\boldsymbol{B}$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to B such that:

- $e(\overline{0}) = \overline{0}^{\boldsymbol{B}}$
- $e(\varphi \to \psi) = e(\varphi) \to^{\boldsymbol{B}} e(\psi) = \neg^{\boldsymbol{B}} e(\varphi) \oplus^{\boldsymbol{B}} e(\psi)$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{\boldsymbol{B}} e(\psi) = \cdots$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{\boldsymbol{B}} e(\psi) = \cdots$

Definition 2.35

A formula φ is a logical consequence of a theory T
w.r.t. a class $\mathbb{K}$ of MV-algebras, $T \models_{\mathbb{K}} \varphi$,
if for every $\boldsymbol{B} \in \mathbb{K}$ and every $\boldsymbol{B}$-evaluation e:

if $e(\gamma) = \overline{1}^{\boldsymbol{B}}$ for every $\gamma \in T$, then $e(\varphi) = \overline{1}^{\boldsymbol{B}}$.

Some theorems and derivations of Ł

Proposition 2.36

(T1) $\vdash_{Ł} \varphi \to \varphi$

(T2) $\vdash_{Ł} \varphi \to (\psi \to \varphi \wedge \psi)$

(D1) $\varphi \leftrightarrow \overline{1} \vdash_{Ł} \varphi$ *and* $\varphi \vdash_{Ł} \varphi \leftrightarrow \overline{1}$

(D2) $\varphi \to \psi \vdash_{Ł} \varphi \wedge \psi \leftrightarrow \varphi$ *and* $\varphi \wedge \psi \leftrightarrow \varphi \vdash_{Ł} \varphi \to \psi$

Proposition 2.37

$\vdash_{Ł} \varphi \oplus \psi \leftrightarrow \psi \oplus \varphi$

$\vdash_{Ł} \varphi \oplus (\psi \oplus \chi) \leftrightarrow (\varphi \oplus \psi) \oplus \chi$

$\vdash_{Ł} \neg(\neg\varphi \oplus \psi) \oplus \psi \leftrightarrow \neg(\neg\psi \oplus \varphi) \oplus \varphi$

$\vdash_{Ł} \overline{0} \oplus \varphi \leftrightarrow \varphi$

$\vdash_{Ł} \varphi \oplus \neg\overline{0} \leftrightarrow \neg\overline{0}$

$\vdash_{Ł} \neg\neg\varphi \leftrightarrow \varphi$

The rule of substitution

Proposition 2.38

$$\vdash_{Ł} \varphi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi \vdash_{Ł} \psi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi, \psi \leftrightarrow \chi \vdash_{Ł} \varphi \leftrightarrow \chi$$

$$\varphi \leftrightarrow \psi \vdash_{Ł} (\varphi \wedge \chi) \leftrightarrow (\psi \wedge \chi) \qquad \varphi \leftrightarrow \psi \vdash_{Ł} (\varphi \vee \chi) \leftrightarrow (\psi \vee \chi)$$

$$\varphi \leftrightarrow \psi \vdash_{Ł} (\chi \wedge \varphi) \leftrightarrow (\chi \wedge \varphi) \qquad \varphi \leftrightarrow \psi \vdash_{Ł} (\chi \vee \varphi) \leftrightarrow (\chi \vee \psi)$$

$$\varphi \leftrightarrow \psi \vdash_{Ł} (\varphi \to \chi) \leftrightarrow (\psi \to \chi) \qquad \varphi \leftrightarrow \psi \vdash_{Ł} (\chi \to \varphi) \leftrightarrow (\chi \to \psi)$$

Corollary 2.39

$\varphi \leftrightarrow \psi \vdash_{Ł} \chi \leftrightarrow \chi'$, *where* χ' *results from* χ *by replacing its subformula* φ *by* ψ*.*

Exercise 11 (Difficult and tedious; but can be automatized)

Prove this corollary and the three previous propositions.

Linear Extensions Property

A theory T is linear if $T \vdash_{Ł} \varphi \to \psi$ or $T \vdash_{Ł} \psi \to \varphi$ for each φ, ψ.

Lemma 2.40 (Linear Extension Property)

If $T \nvdash_{Ł} \varphi$, then there is linear theory $T' \supseteq T$ s.t. $T' \nvdash_{Ł} \varphi$.

The proof is the same as in the case of Gödel–Dummett logic using the Semilinearity property we have proved in the previous section.

Lindenbaum–Tarski algebra

Definition 2.41

Let T be a theory. We define

$$[\varphi]_T = \{\psi \mid T \vdash_{Ł} \varphi \leftrightarrow \psi\} \qquad L_T = \{[\varphi]_T \mid \varphi \in Fm_{\mathcal{L}}\}$$

The Lindenbaum–Tarski algebra of a theory T ($\mathbf{LindT}_T$) as an algebra with the domain L_T and operations:

$$\overline{0}^{\mathbf{LindT}_T} = [\overline{0}]_T$$
$$\neg^{\mathbf{LindT}_T}[\varphi]_T = [\varphi \to \overline{0}]_T$$
$$[\varphi]_T \oplus^{\mathbf{LindT}_T} [\psi]_T = [\neg\varphi \to \psi]_T$$

Lindenbaum–Tarski algebra: basic properties

Proposition 2.42

1. $[\varphi]_T = \overline{1}^{\mathbf{LindT}_T}$ *iff* $T \vdash_{Ł} \varphi$.
2. $[\varphi]_T \leq [\psi]_T$ *iff* $T \vdash_{Ł} \varphi \to \psi$.
3. $\mathbf{LindT}_T$ *is an* MV*-algebra.*
4. $\mathbf{LindT}_T$ *is an* MV*-chain iff* T *is linear.*

Proof.

The same as in the case of Gödel–Dummett logic we only use Proposition 2.37 to prove 3. □

Three completeness theorems

Theorem 2.43

The following are equivalent for every theory T and a formula φ:

1. $T \vdash_{Ł} \varphi$
2. $T \models_{\mathbb{MV}} \varphi$ — *w.r.t. general semantics*
3. $T \models_{\mathbb{MV}_{lin}} \varphi$ — *w.r.t. linear semantics*

If T is finite we can add:

4. $T \models_{[0,1]_{Ł}} \varphi$ — *w.r.t. standard semantics*

Exercise 12 (Easy)

Prove the equivalence of the first three claims.

We give a proof of 3. implies 4. but first . . .

MV-algebras and LOAGs

A lattice ordered Abelian group (*LOAG* for short) is a structure $\langle G, +, 0, -, \leq\rangle$ s.t. $\langle G, +, 0, -\rangle$ is an Abelian group and:

(i) $\langle G, \leq\rangle$ *is a lattice,*
(ii) *if* $x \leq y$*, then* $x + z \leq y + z$ *for all* $z \in G$*.*

A strong unit u is an element s.t.

$$(\forall x \in G)(\exists n \in N)(x \leq nu)$$

For LOAG $\boldsymbol{G} = \langle G, +, 0, -, \leq\rangle$ and strong unit u we define algebra $\Gamma(\boldsymbol{G}, u) = \langle [0, u], \oplus, \neg, \overline{0}\rangle$, where $x \oplus y = \min\{u, x + y\}$, $\neg x = u - x$, $\overline{0} = 0$.

By $\boldsymbol{R}$ we denote the additive LOAG of reals.

Proposition 2.44

$\Gamma(\boldsymbol{G}, u)$ *is an* MV*-algebra and for each* $u > 0$ *is* $\Gamma(\boldsymbol{R}, u)$ *isomorphic to the standard* MV*-algebra* $[0, 1]_{Ł}$*.*

Proof of std. completeness of Łukasiewicz logic

If $T \nvdash_{\mathrm{Ł}} \varphi$ we know that there is countable MV-chain $\boldsymbol{B}$ s.t. $T \not\models_{\boldsymbol{B}} \varphi$. Let $x_1, \ldots, x_n$ be variables occurring in $T \cup \{\varphi\}$. Then:

$$\not\models_{\boldsymbol{B}} (\forall x_1, \ldots, x_n) \bigwedge_{\psi \in T} (\psi \approx \overline{1}) \Rightarrow (\varphi \approx \overline{1})$$

Let us define algebra $\boldsymbol{B}' = \langle Z \times B, +, -, 0 \rangle$ as:

$$\langle i, x \rangle + \langle j, y \rangle = \begin{cases} \langle i + j, x \oplus y \rangle & \text{if } x \,\&\, y = 0 \\ \langle i + j + 1, x \,\&\, y \rangle & \text{otherwise} \end{cases}$$

$$-\langle i, x \rangle = \langle -i - 1, \neg x \rangle \text{ and } 0 = \langle 0, \overline{0} \rangle$$

Proposition 2.45

$\boldsymbol{B}'$ is a LOAG and $\boldsymbol{B} = \Gamma(\boldsymbol{B}', \langle 1, \overline{0} \rangle)$.

Proof of std. completeness of Łukasiewicz logic

Let us fix an extra variable u, we define a translation of MV-terms into LOAG-terms:

$$x' = x \qquad \overline{0}' = 0 \qquad (\neg t)' = u - t' \qquad (t_1 \oplus t_2)' = (t_1' + t_2') \wedge u.$$

Recall that we have:

$$\not\models_{\boldsymbol{B}} (\forall x_1, \dots, x_n) \bigwedge_{\psi \in T} (\psi \approx \overline{1}) \Rightarrow (\varphi \approx \overline{1}),$$

Thus also:

$$\not\models_{\boldsymbol{B}'} (\forall u)(\forall x_1, \dots, x_n)[(0 < u) \wedge \bigwedge_{i \leq n} (x_i \leq u) \wedge (0 \leq x_i) \wedge \bigwedge_{\psi \in T} (\psi' \approx \overline{1}) \Rightarrow (\varphi' \approx \overline{1})$$

Proof of std. completeness of Łukasiewicz logic

Gurevich–Kokorin theorem: each $\forall_1$-sentence of LOAGs is true in additive LOAG of reals iff it is true in all linearly ordered LOAGs. Thus

$$\not\models_{\boldsymbol{R}} (\forall u)(\forall x_1,\dots,x_n)[(0<u)\wedge\bigwedge_{i\leq n}(x_i\leq u)\wedge(0\leq x_i)\wedge\bigwedge_{\psi\in T}(\psi'\approx\overline{1})\Rightarrow(\varphi'\approx\overline{1})]$$

And so

$$\not\models_{\Gamma(\boldsymbol{R},u)} (\forall x_1,\dots,x_n)\bigwedge_{\psi\in T}(\psi\approx\overline{1})\Rightarrow(\varphi\approx\overline{1})$$

And so

$$\not\models_{[0,1]_{\text{Ł}}} (\forall x_1,\dots,x_n)\bigwedge_{\psi\in T}(\psi\approx\overline{1})\Rightarrow(\varphi\approx\overline{1})$$

i.e., $T\not\models_{[0,1]_{\text{Ł}}}\varphi$

What is a logic? (as a mathematical object) (and for us here)

Convention

A logic is a provability relation on formulae in a language $\mathcal{L} \supseteq \{\to, \vee, \overline{1}\}$ axiomatized by axioms axioms $\mathcal{A}x$ and rules $\mathcal{R}u$ s.t.

- $\vdash_{\mathrm{L}} \varphi \to \varphi \qquad \varphi, \varphi \to \psi \vdash_{\mathrm{L}} \psi \qquad \varphi \to \psi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \to \chi$
- $\varphi \vdash_{\mathrm{L}} \overline{1} \to \varphi \qquad \overline{1} \to \varphi \vdash_{\mathrm{L}} \varphi$
- $\vdash_{\mathrm{L}} \varphi \to \varphi \vee \psi \qquad \vdash_{\mathrm{L}} \psi \to \varphi \vee \psi \qquad \varphi \to \chi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \vee \psi \to \chi$
- for each n-ary connective $c \in \mathcal{L}$, $\mathcal{L}$-formulae $\varphi, \psi, \chi_1, \dots, \chi_n$, and each $i < n$ the following holds:

  $$\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{L}} c(\chi_1, \dots, \chi_i, \varphi, \dots, \chi_n) \leftrightarrow c(\chi_1, \dots, \chi_i, \psi, \dots, \chi_n)$$

- each of the rules has only finitely many premises

We fix a logic L in language $\mathcal{L}$ with axioms $\mathcal{A}x$ and rules $\mathcal{R}u$.

Semantical consequence w.r.t. a class of $\mathbf{G}$-algebras

Definition 2.46

A ***B*-evaluation** is a mapping e from $Fm_{\mathcal{L}}$ to B such that:

- $e(\overline{0}) = \overline{0}^{\boldsymbol{B}}$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{\boldsymbol{B}} e(\psi)$
- $e(\varphi \to \psi) = e(\varphi) \to^{\boldsymbol{B}} e(\psi)$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{\boldsymbol{B}} e(\psi)$

Definition 2.47

A formula φ is a logical consequence of a theory T w.r.t. a class $\mathbb{K}$ of $\mathbf{G}$-algebras, $T \models_{\mathbb{K}} \varphi$, if for every $\boldsymbol{B} \in \mathbb{K}$ and every $\boldsymbol{B}$-evaluation e:

if $e(\gamma) = \overline{1}^{\boldsymbol{B}}$ for every $\gamma \in T$, then $e(\varphi) = \overline{1}^{\boldsymbol{B}}$.

Semantical consequence w.r.t. a class of $\mathcal{L}$-algebras

Definition 2.48

A $\boldsymbol{B}$-evaluation is a mapping e from $Fm_{\mathcal{L}}$ to B such that:

- $e(\overline{1}) = \overline{1}^{\boldsymbol{B}}$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{\boldsymbol{B}} e(\psi)$
- $e(\varphi \to \psi) = e(\varphi) \to^{\boldsymbol{B}} e(\psi)$
- $e(c(\chi_1, \dots, \chi_n)) = c^{\boldsymbol{B}}(e(\chi_1), \dots, e(\chi_n))$ for each n-ary $c \in \mathcal{L}$

Definition 2.49

A formula φ is a logical consequence of a theory T w.r.t. a class $\mathbb{K}$ of $\mathcal{L}$-algebras, $T \models_{\mathbb{K}} \varphi$, if for every $\boldsymbol{B} \in \mathbb{K}$ and every $\boldsymbol{B}$-evaluation e:

if $e(\gamma) \vee^{\boldsymbol{B}} \overline{1}^{\boldsymbol{B}} = e(\gamma)$ for every $\gamma \in T$, then $e(\varphi) \vee^{\boldsymbol{B}} \overline{1}^{\boldsymbol{B}} = e(\varphi)$.

Algebraic semantics and semilinear logic

For each L there is a class $\mathbb{L}$ of L-algebras st. for every theory T and a formula φ we have:

$$T \vdash_{\mathrm{L}} \varphi \text{ if, and only if, } T \models_{\mathbb{L}} \varphi.$$

Each L-algebra $\boldsymbol{A}$ can be ordered:

$$x \leq_{\boldsymbol{A}} y \quad \text{IFF} \quad x \vee^{\boldsymbol{A}} y = y \quad \text{IFF} \quad \overline{1}^{\boldsymbol{A}} \leq x \rightarrow^{\boldsymbol{A}} y$$

Let us by $\mathbb{L}_{\mathrm{lin}}$ of L-algebras which are linearly ordered

Definition 2.50

We say that a logic L is semilinear if for every theory T and a formula φ we have:

$$T \vdash_{\mathrm{L}} \varphi \text{ if, and only if, } T \models_{\mathbb{L}_{\mathrm{lin}}} \varphi.$$

Syntactical characterization of semilinearity

Theorem 2.51 (Syntactical characterization)

Let L *be axiomatized by axioms* Ax *and rules* $\mathcal{R}u$. *TFAE:*

1. L *is a semilinear logic*
2. *If* $T \nvdash_{\mathrm{L}} \varphi$ *then there is a linear theory* $S \supseteq T$ *s.t.* $S \nvdash_{\mathrm{L}} \varphi$
3. *For every set of formulae* $T \cup \{\varphi, \psi, \chi\}$*:*
$$T, \varphi \to \psi \vdash_{\mathrm{L}} \chi \quad \textit{and} \quad T, \psi \to \varphi \vdash_{\mathrm{L}} \chi \quad \textit{imply} \quad T \vdash_{\mathrm{L}} \chi.$$
4. $\vdash_{\mathrm{L}} (\varphi \to \psi) \vee (\psi \to \varphi)$ *and for every set of formulae* $T \cup \{\varphi, \psi, \chi\}$*:*
$$T, \varphi \vdash_{\mathrm{L}} \chi \quad \textit{and} \quad T, \psi \vdash_{\mathrm{L}} \chi \quad \textit{imply} \quad T, \varphi \vee \psi \vdash_{\mathrm{L}} \chi.$$
5. $\vdash_{\mathrm{L}} (\varphi \to \psi) \vee (\psi \to \varphi)$ *and if* $T \vdash_{\mathrm{L}} \varphi$, *then* $T \vee \chi \vdash_{\mathrm{L}} \varphi \vee \chi$ *for all* χ*s*
6. $\vdash_{\mathrm{L}} (\varphi \to \psi) \vee (\psi \to \varphi)$ *and if* $T \vdash \varphi \in \mathcal{R}u$, *then*
$$T \vee \chi \vdash_{\mathrm{L}} \varphi \vee \chi \textit{ for all } \chi\textit{s}$$

Semantical characterization of semilinearity

Theorem 2.52 (Semantic characterization)

Let L *be a logic. TFAE:*

1. L *is a semilinear logic*
2. *the finitely relatively subdirectly irreducible* L*-algebras are exactly the* L*-chains*
3. *the relatively subdirectly irreducible* L*-algebras are linearly ordered*

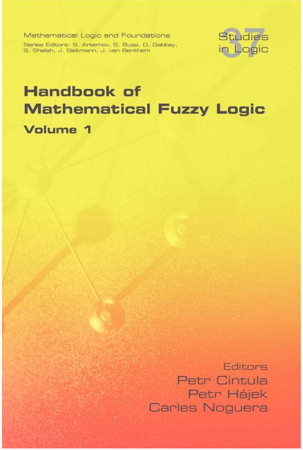
The moral of the story . . .

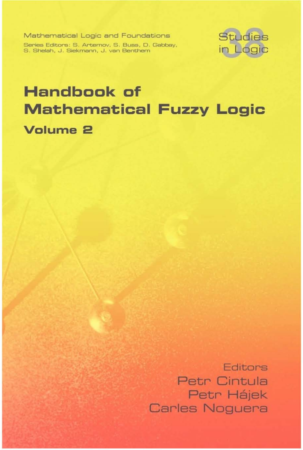
1. fuzzy logics are not so different from the classical logic, they have
   - Hilbert style axiomatizations
     (and even analytic proof system based on the hypersequents)
   - semantics based on real numbers or (linearly) ordered algebras
   - a completeness theorem linking those two facets
   - usually a co-NP-complete set of theorems (e.g. Łukasiewicz or G)
2. but there are funny things going on:
   - deduction theorem could fail
   - compactness and finitarity are two different notions

   ⋮

3. numerous different fuzzy logics can be designed playing with
   the axiomatization or the semantics

If you want to know more . . .





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