Is truth gradable?
From Bivalence to Many-Valued Logics

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Carolina, born 2009. Picture taken Summer 2014.

“Carolina is a tall child.”

"Carolina is a little plump."

"Carolina's hair is blonde."

"Carolina's hair is not red."

"Carolina is taller than she is plump."

"Carolina will eat an ice-cream tomorrow."

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Is this true or false — *now*? One way to analyse Aristotle's example from a logical point of view is to use modal logic, in the specific form of temporal (a.k.a. tense) logic. Similarly for:

> *Carolina will eat an ice cream tomorrow.*

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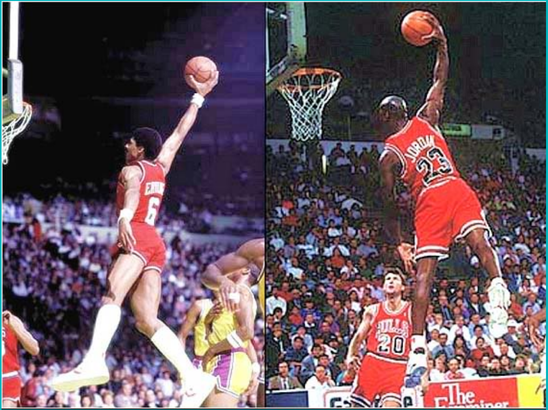
Consider, as a case in point, gradable predicates: predicates that admit comparatives. Such is, e.g., the monadic predicate Tall. Not such is, e.g., the binary predicate Equal:

> *All animals are equal, but some animals are more equal than others.*
>
> G. Orwell, *Animal Farm*, 1945

It's easy to be puzzled by gradable predicates such as Tall.



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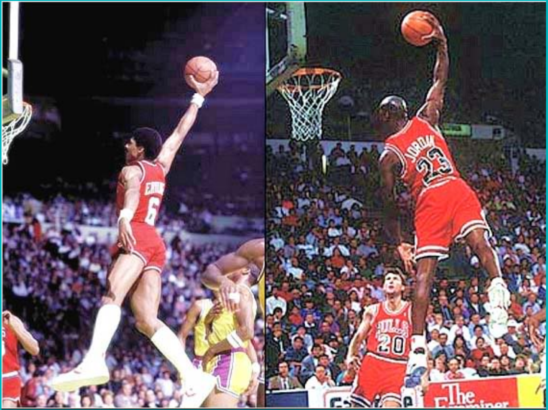

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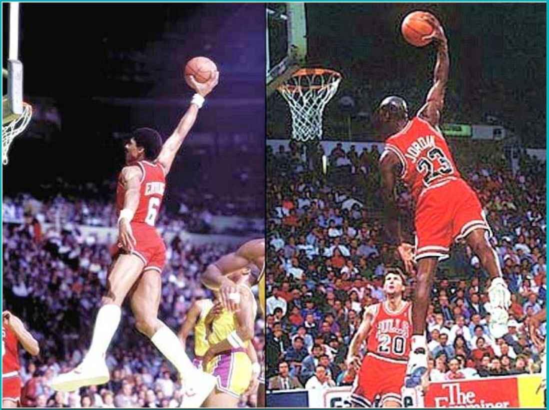

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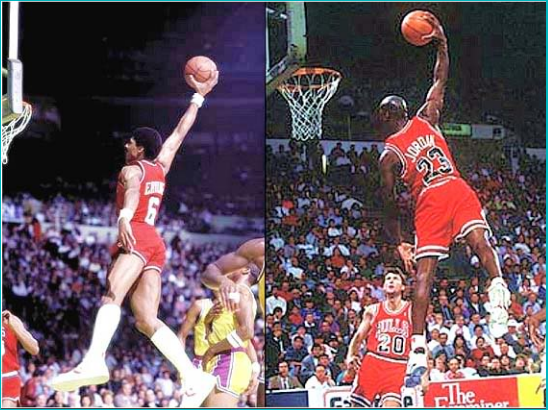

Julius Erving (Dr. J.) Michael Jordan (Air)

1. It is true that *"Erving is tall"*.
2. It is true that *"Jordan is tall"*.
3. It is true that *"Erving is taller than Jordan."*

It seems plain that statements such as:

"Carolina is a lot taller than she is plump"

or, essentially equivalently,

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convey some sort of information. Likewise, it is clear that some sort of comparison is going on.

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One source of problems in tackling these questions is that the predicates Tall and Plump are vague. To moderate our ambitions, we shall now restrict attention to predicates that are non-classical on account of their being vague.

Jan Łukasiewicz 1878–1956

Vaguely Speaking

(A prologue)

The Sorites Paradox

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- *…*
- *If* (10^{100}-1) *grains of wheat do not make a heap, then* 10^{100} *do not.*
- *Hence:* 10^{100} *grains of wheat do not make a heap.*

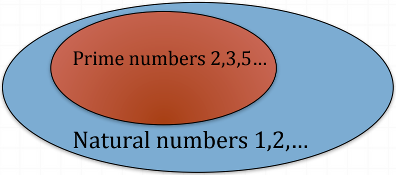
The Sorites Paradox

- Modern response: *Theories of Vagueness.*
- Initial problem: the monadic predicate `Heap(x)` is vague.
- To explain the paradox away we need a theory of such vague predicates.
- Any such theory needs some pre-theoretical, or at least theory-neutral, understanding of what a "vague predicate" is.
- Building on such a common pre-theoretical understanding of vagueness, a plethora of conflicting theories of vagueness has been advanced in the 20th century.
- So there is no explanation of the Sorites Paradox that is "standard", in the sense of being most widely accepted.

Theory-neutral features of vagueness

Features of a precise predicate.

The monadic predicate $P(x)$:="x is prime", interpreted over the set of natural numbers $x \geq 1$, is (absolutely) *precise*: its extension is the set of prime numbers; its anti-extension is the set of composite numbers; each number either belongs to the extension of P or to its anti-extension, but not to both; and in principle there is no issue as to whether a given number be prime or composite — though in practice it may be impossible to ascertain which is the case for an astronomic instance of x.



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By contrast, the monadic predicate $R(x) :=$"x is red", interpreted over the set of all objects, is (to some extent) *vague*: its extension ought to be the set of all red objects; its anti-extension ought to be the set of all non-red objects; but it may not be clear, even in principle, just which objects do qualify as red, and which as non-red — think of a peculiar tint at the borderline between red and pink.

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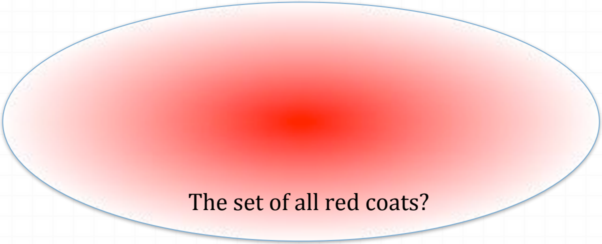

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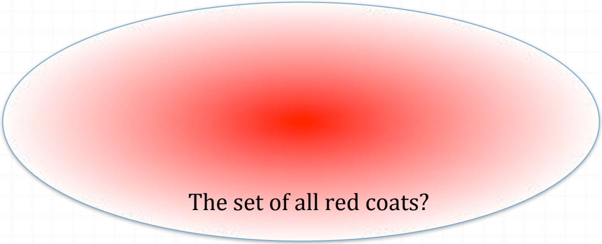

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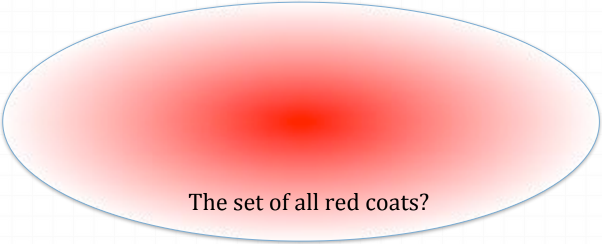


Theory-neutral features of vagueness

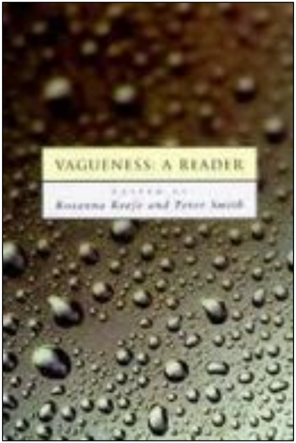
Features of a (monadic) vague predicate *R*:

(FV1) R admits *borderline cases* over the intended domain of interpretation D, *i.e.* there are instantiations of $R(x)$ by (a term naming a constant) $c \in D$ such that it is unclear whether $R(c)$ holds or its negation $\neg R(c)$ does.

(FV2) R lacks *sharp boundaries* over the intended domain of interpretation D, *i.e.* there is no clearly defined boundary separating the extension of $R(\cdot)$ from its anti-extension.

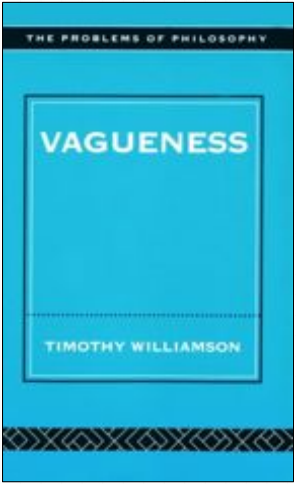
(FV3) R is susceptible to a *Sorites series* over the intended domain of interpretation D, *i.e.* there are instantiations of $R(x)$ by $c_1, \ldots, c_n \in D$ such that it is clear that $R(c_1)$ holds, it is clear that $R(c_n)$ does not hold, and it seems at least plausible that if $R(c_i)$ holds then so does $R(c_{i+1})$, for each $i \in \{1, \ldots, n-1\}$.

Theories of vagueness



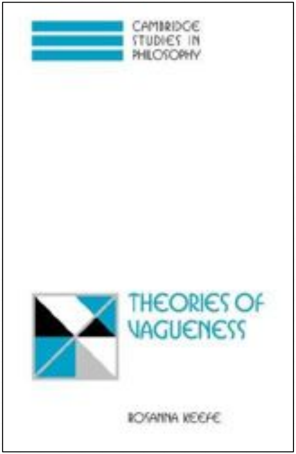
Useful Reader: R. Keefe and P. Smith, eds.

Theories of vagueness



Epistemicism: Vagueness as Ignorance

Theories of vagueness



Supervaluationism: Vagueness as Precisifiability

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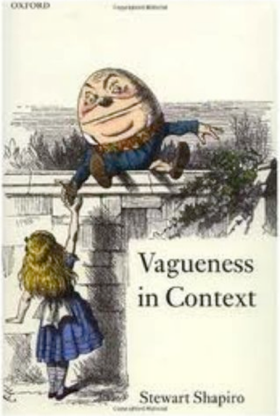
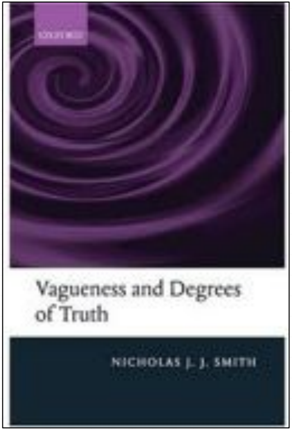


Contextualism: Vagueness as dependence from Context

Theories of vagueness



Degree-Based Theories: Vagueness as Truth-in-Degrees

Degree-Based Theories of Vagueness

Main Assumption: Truth comes in degrees.

- If x is a clear case of R, then $R(x)$ is (fully, classically) true.
- If x is a clear non-case of R, then $R(x)$ is (fully, classically) false.
- If x is a borderline case of R, then $R(x)$ is true (or false) **to a degree**.

It may seem natural to say that, in borderline cases, a certain coat is neither clearly red, nor clearly non-red, so that "*This coat is red*" is **neither true nor false**. And the further step of then saying that "*This coat is red*" is true (or false) **to some degree** may also sound appealing. (Well, does it sound appealing to you?) But we should be aware that taking this direction is a **major departure from the roots of logic** as we know it, both philosophically and mathematically.

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Frege on Truth

We are therefore driven into accepting the truth value [*Wahrheitswert*] of a sentence as constituting its reference [*Bedeutung*]. By the truth value of a sentence I understand the circumstance that it is true or false. There are no further truth values. For brevity I call the one the True [*das Wahre*], the other the False [*das Falsche*].

G. Frege, *On Sense and Reference*, 1892, p. 34.

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In other writings (notably the unpublished *Logik*), Frege makes the following very clear.

- Truth is a primitive notion in logic: it cannot be defined.
- `True(`p`)` is a peculiar predicate in that it does not admit comparatives: p is truer than q is a *façon de parler* lacking genuine logical content.
- (Implicitly.) In particular, **degrees of truth are non-sense**, according to Fregean orthodoxy.

Main objections to degree-theoretic accounts of vagueness:

1. Compositionality (K. Fine, 1975).
2. Higher-order vagueness (T. Williamson *et al.*, 1994).
3. Artificial precision (R. Keefe *et al.*, 2000).

I ignore higher-order vagueness here, for brevity. I report Fine's arguments against compositionality, and those of Keefe *et al.* on artificial precision. For further information on artificial precision and related issues:

- V.M., *The problem of artificial precision in theories of vagueness: the rôle of maximal consistency*, Erkenntnis, 2014.
- V.M., *Is there a probability theory of many-valued events?*, in Probability, uncertainty and rationality, CRM Series, 10, Ed. della Normale, Pisa, 2010.

K. Fine, *Vagueness, Truth and Logic*, Synthese, 1975.

KIT FINE

VAGUENESS, TRUTH AND LOGIC[1]

This paper began with the question 'What is the correct logic of vagueness?' This led to the further question 'What are the correct truth-conditions for a vague language?', which led, in its turn, to a more general consideration of meaning and existence. The first half of the paper contains the basic material. Section 1 expounds and criticizes one approach to the problem of truth-conditions. It is based upon an extension of the standard truth-tables and falls foul of something called penumbral connection. Section 2 introduces an alternative framework, within which penumbral connection can be accommodated. The key idea is to consider not only the truth-values that sentences actually receive but also the truth-values that they might receive under different ways of making them more precise. Section 3 describes and defends the favoured account within this framework. Very roughly, it says that a vague sentence is true if and only if it is true for all ways of making it completely precise. The second half of the paper deals with consequences, complications and comparisons. Sec-

Is any account along truth-value lines acceptable? Any account that satisfies the conditions F and S would always appear to make correct allocations of definite truth-value. However, even the maximizing policy fails to make many correct allocations of definite truth-value. For suppose that a certain blob is on the border of pink and red and let P be the sentence 'the blob is pink' and R the sentence 'the blob is red'. Then the conjunction P & R is false since the predicates 'is pink' and 'is red' are contraries. But on the maximizing account the conjunction P & R is indefinite since both of the conjuncts P and R are indefinite.

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The specific examples chosen should not blind us to the general point that they illustrate. It is that logical relations may hold among predicates with borderline cases or, more generally, among indefinite sentences. Given the predicate 'is red', one can understand the predicate 'is non-red' to be its contradictory: the boundary of the one shifts, as it were, with the boundary of the other. Indeed, it is not even clear that convincing examples require special predicates. Surely $P \,\&\, -P$ is false even though P is indefinite.

Let us refer to the possibility that logical relations hold among indefinite sentences as *penumbral connection*; and let us call the truths that arise, wholly or in part, from penumbral connection, *truths on a penumbra* or *penumbral truths*. Then our argument is that no natural truth-value approach respects penumbral truths. In particular, such an approach cannot distinguish between 'red' and 'pink' as independent and as exclusive upon their common penumbra.

Fine's argument about penumbral connection is considered by most as a definitive objection to any attempt of regarding a (truth-functional) many-valued logic as a formalisation of the logic of vague propositions. See e.g. Williamson's treatise on this point.

Artificial precision

[Fuzzy logic] imposes artificial precision [... While] one is not obliged to require that a predicate either definitely applies or definitely does not apply, one is obliged to require that a predicate applies to such-and-such, rather than to such-and-such other, degree (e.g. that a man 5ft 10in tall belongs to tall *to degree* 0.6 *rather than 0.5).*

S. Haack, 1979

Artificial precision

The degree theorist's assignments impose precision in a form that is just as unacceptable as a classical true/false assignment. [. . .] *All predications of "is red" will receive a unique, exact value, but it seems inappropriate to associate our vague predicate "red" with any particular exact function from objects to degrees of truth. For a start, what could determine which is the correct function, settling that my coat is red to degree 0.322 rather than 0.321?*

R. Keefe, 2000

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One general response, however, is tempting, and is periodically readvanced in the literature:

Response: The truth value of $P(x)$ is simply the normalised measurement of the physical observable that underlies the predicate P. (Normalisation is assumed to be linear, as usual.)

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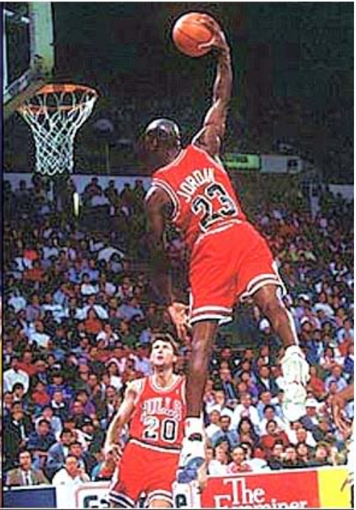
Example: The truth value of $p :=$"Enzo is tall" is the height of Enzo (=190cm) linearly renormalised over $[0, 1]$.

Taking e.g. as maximum height 250cm, and as minimum height 90cm, the truth value of p is $\frac{190-90}{250-90} = \frac{100}{160} = 0.625$.

Rebuttal:



Julius Erving Michael Jordan

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is untenable. Consider the predicate Tall, written $T(x)$. Then:

$$\text{It is the case that } T(\text{Jordan}). \tag{$\star$}$$

Also,

$$\text{It is the case that } T(\text{Erving}). \tag{$*$}$$

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Let $H(x) \in [0,1]$ denote a bijective order-preserving renormalisation of the measured heights of individuals in a given domain. By $(\star)$, $H(\text{Jordan}) = 1$. By $(*)$, $H(\text{Erving}) = 1$.

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By $(\dagger)$ and our assumptions on the renormalisation map H,

$$H(\text{Jordan}) = 1 < 1 = H(\text{Erving}), \quad \text{contradiction.}$$

There also is one **counter-response** to this rebuttal that is relatively common, which is however mistaken:

- X :=“Jordan is tall”.
- Y :=“Erving is tall”.
- There is now no problem with $w(X) < w(Y)$, as X and Y are distinct propositional variables. Nothing forces $w(X) = w(Y)$.

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- X :="Jordan is tall".
- Y :="Erving is tall".
- There is now no problem with $w(X) < w(Y)$, as X and Y are distinct propositional variables. Nothing forces $w(X) = w(Y)$.

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This propositional "solution" amounts to considering two distinct predicates related to tallness: one for the height of Erving, and one for the height of Jordan.

But the point of predicate logic is precisely that predicates can be applied to a variety of terms: fixing the context etc. there should be one predicate $T(x)$ for "x is tall"—lest there be no logic at all. The counter response is worse than the original problem: it leads us to reject the possibility that there is a logic of such a monadic predicate as Tall.

Clear Assumptions about Vague Predicates

(Blanket) Assumption 0

Truth *is* gradable. The (Fregean) *denotatum* of a proposition involving (vague) predicates is its degree of truth. Degrees of truth may be universally compared, i.e., they form a totally ordered set.

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This is at one and the same time a very Fregean — hence classical — assumption, in that we are postulating the existence of referents of propositions in the Fregean sense, and a very anti-Fregean assumption, in that we are postulating the existence of a degrees of truth (which moreover are totally ordered).

We now add a set of more specific assumptions, with the intent of identifying at least *one* formal system which may be a model for the logic of *some* vague predicates. Later we shall revert to the sole Assumption 0 in search of a more systematic treatment of gradable truth.

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Assumption I

Each vague predicate has a well-defined extension.

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Assumption I

Each vague predicate has a well-defined extension.

This does not entail that the predicate is precise, or that it does not admit borderline cases, etc. Indeed, given any x, it is a matter of classical logic that:

- Either it is the case that $\mathsf{Tall}(x)$, i.e. x is a clear, indisputable case of a tall individual;
- Or it is not the case that $\mathsf{Tall}(x)$, i.e. x is not a clear, indisputable case of a tall individual.

Consequently, one cannot assert a vague predicate tentatively, or to a degree.

In the *Begriffsschrift*, Frege introduced the sign $\vdash$ as a compound formation:

$-$	the *content* stroke
$\mid$	the *judgement* stroke
$\vdash$	the *assertion* sign
$\vdash \alpha$	means: α (assertion of).

Hence, by

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we mean: (it is asserted that) x is a clear, indisputable case of tallness.

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Comment. There are formal systems, such as Pavelka's logic, where inference is indexed by a degree. But it is unclear whether one can make sense at all of the idea of "asserting (or assuming) a proposition to a degree", and even less of the idea of "deducing α from β to a degree".

Assumption II

We only consider vague predicates admitting an antonym.

E.g., Tall–Short, Near–Far, etc.

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Assumptions I & II directly lead to 3 notions of negation:

Predicate	Extension
$-$Tall	Set-theoretic complement of the extension of Tall
$\neg$Tall	Extension of the opposite predicate Short
$\sim$Tall	Extension of the predicate non-Tall

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Predicate	Meaning
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Assumption III

We only consider the negation connective $\neg$.

Tall and Red are fundamentally different vague predicates.

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- Tall has a natural antonymic, or opposite, or contrary predicate, namely, Short. In symbols,

$$\neg \mathsf{Tall}(x) \equiv (\neg \mathsf{Tall})(x) \equiv \mathsf{Short}(x).$$

Similarly: Young, Beautiful, etc.

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- Red does not have a natural contrary. There is no name for opposite-to-Red in the colour spectrum. Similarly: Cute, Nice, etc. Hence:

$\neg$Red just doesn't make sense.

The distinction above is strictly a matter of logic, not of linguistic usage or what have you.

The negation $-$ must obey the Double Negation Law

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Indeed, — behaves like a classical negation:

- The extension of Tall is the set of individuals which are a clear, indisputable case of tallness.

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- The extension of —Tall is the set of individuals which are not a clear, indisputable case of tallness.
- Hence, the extension of —(—Tall) coincides with the extension of Tall: set-theoretic complement is idempotent.

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- The extension of $\neg$Tall is the set of individuals which are a clear, indisputable case of shortness.

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- The extension of $\neg$Tall is the set of individuals which are a clear, indisputable case of shortness.
- Hence, the extension of $\neg(\neg$Tall$)$ coincides with the extension of Tall: the antonym of the antonym of Tall is Tall.

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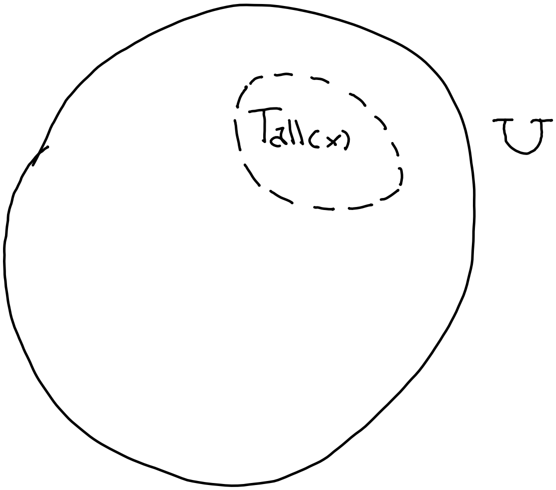
Indeed, $\sim$ behaves like an Intuitionistic pseudo-complement:

- The extension of Red is the set of objects which are a clear, indisputable case of redness.
- The extension of $\sim$Red is the set of objects which are a clear, indisputable case of non-redness.
- Hence, the extension of $\sim(\sim\text{Red})$ is the set of objects which do not qualify as a clear case of non-redness; but in general they will not qualify as a clear case of redness, either.


Tall(x)
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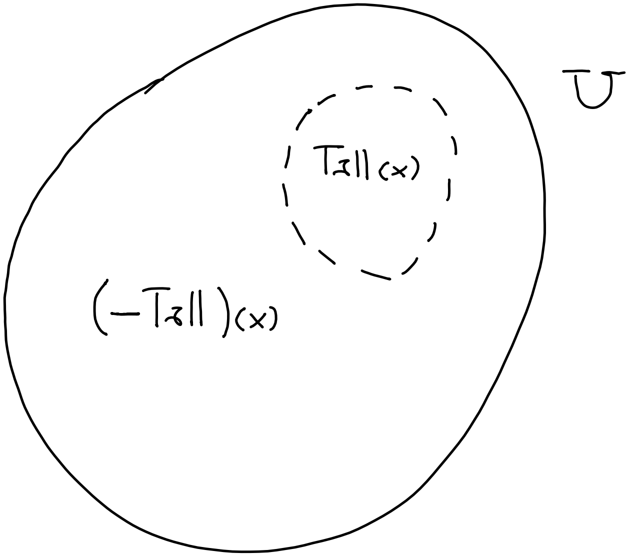
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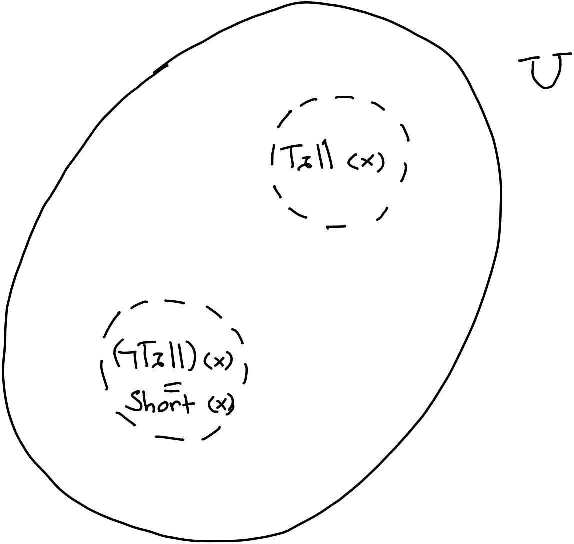
(¬Tall)(x)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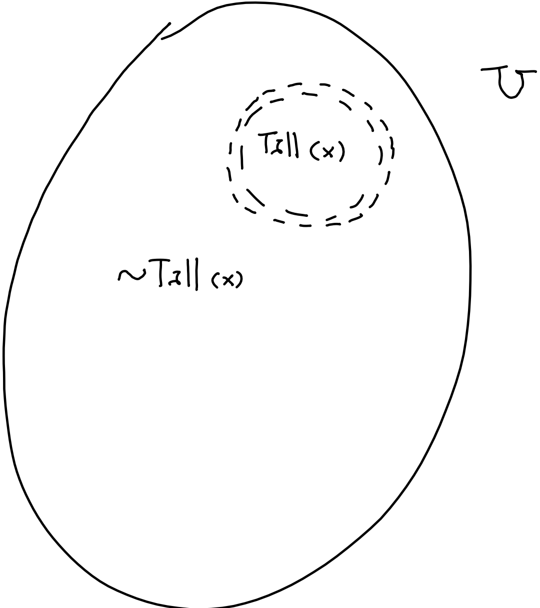
Tall (x)
(¬Tall) (x)
=
Short (x)



U
Tall (x)
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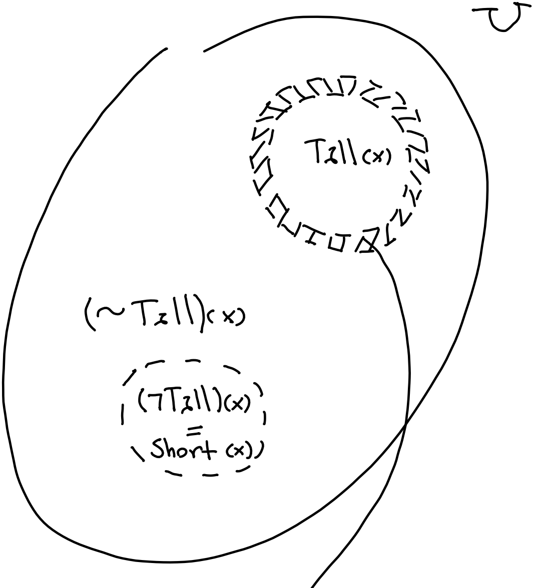
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Let us take stock:

- $\neg$Tall applies to anything that is clearly opposite to tall, i.e. is clearly short.
- $\neg$Red just doesn't make sense, because there is no opposite to redness.

We henceforth restrict attention to predicates such as Tall, which admit of antonyms such as $\neg\mathsf{Tall} \equiv \mathsf{Short}$. We only consider the negation $\neg$.

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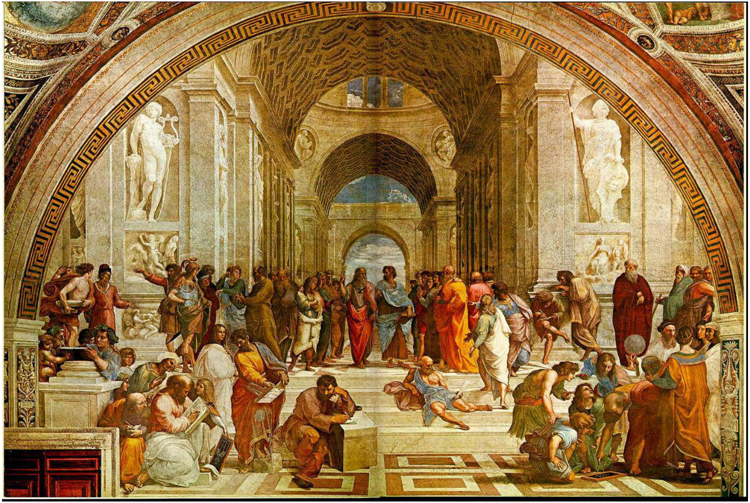
We henceforth restrict attention to predicates such as Tall, which admit of antonyms such as ¬Tall ≡ Short. We only consider the negation ¬.

We have made some assumptions about a unary connective, negation. The next key issue now is:

What binary connectives are basic for vague predicates?

True, Truer, Much Truer



Raffaello Sanzio, La Scuola di Atene, *ca. 1509.*

Aristotle's example, in the *Topics*, of inference with *comparatives of comparatives*.

P1. x is more T than z.

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For instance:

P1. Ada is more tall than Carolina.

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P2. Blaise is more tall than Carolina.

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C. Ada is more tall than Blaise.

Note that as we step up from the subsentential to the sentential level, to model the Aristotelian

$$x \text{ is more } T \text{ than } z$$

with a *single connective independent of T*, it seems unavoidable to move on to the sentence

$$T(x) \text{ is more true than } T(z).$$

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Key Fact

The comparison connective

$$\alpha \text{ is more true than } \beta$$

produces a classical proposition out of the given α and β.

Hence the connective is more true than cannot play a fundamental rôle in the logic of vague predicates.

A Standard Mistake

Many-valued logics (after Hájek) are logics of comparative truth wherein the implication connective

$$\alpha \to \beta$$

is read

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This is simply untenable.

A Standard Mistake

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Proper Reformulation

Many-valued logics (after Hájek) are logics of comparative truth wherein the assertion

$$\vdash \alpha \longrightarrow \beta$$

is read

$$\text{It is the case that } \alpha \text{ is less true than } \beta.$$

Standard account of meaning of a proposition/predicate as its truth conditions:

> [...] *to grasp a thought is to know the conditions for it to be true.*
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E.g., you know what $\mathsf{Prime}(x)$ means as soon as you can tell a prime number when you see it. Compare:

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Just Wrong $(\star)$

You know what $\mathsf{Tall}(x)$ means as soon as you can tell a (clearly, indisputably) tall person when you see one.

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E.g., you know what Prime(x) means as soon as you can tell a prime number when you see it. Compare:

Just Wrong ($\star$)

You know what Tall(x) means as soon as you can tell a (clearly, indisputably) tall person when you see one.

For, what about a tallish, though not indisputably tall person? And even an indisputably short person? You may perfectly meet ($\star$) and yet be completely in the dark as to whether a clear, indisputable case of a short person indeed is short. That's no grasping of Tall(x), on any sensible account.

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Example

In Intuitionistic logic — and in Gödel-Dummett logic — the Lindenbaum-Tarski equivalence class of any proposition α is uniquely determined by the collection of (intuitionistic) valuations that make α true.

Mathematically, this is precisely why in Intuitionistic logic, like in classical logic, one can develop Stone-Esakia-Priestley duality for Heyting algebras in the extensional language of *clopen upper sets*, and give up the intensional language of functions, i.e. the Fregean "courses of values". We will see that in the logic of vague predicates we are seeking to pin down, which will turn out to be Łukasiewicz' logic, no such simple extensional reduction is possible.

So what can we resort to if "more true than" won't do? Aristotle's example, in the *Topics*, of inference with *comparatives of comparatives*.

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A propositional translation of Aristotle's example:

P1. $T(x)$ is more true than $T(z)$.

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$\alpha \rhd \beta$	means: *α is much more true than β*

For example,

$$\mathsf{Tall}(x) \rhd \mathsf{Young}(y)$$

means:

x is much more a case of tallness than y is a case of youth.

$\alpha \rhd \beta$	means: α *is much more true than* β

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Assumption IV

All vague propositions/predicates may be combined through $\rhd$ to yield new compound vague propositions/predicates.

In particular, this means that $\alpha \rhd \beta$ has again a well-defined extension, an antonym, and a well-defined anti-extension (Assumptions I–III).

Assumption V (Truth conditions of $\rhd$)

$$\vdash \alpha \rhd \beta \qquad \text{if, and only if,} \qquad \vdash \alpha \text{ and } \vdash \neg\beta.$$

Consider the sentence

Frege is much more intelligent than he is handsome.

This is a vague proposition of the from $\alpha \rhd \beta$. What does one mean when one asserts it, i.e. when

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Recalling our Assumption I, the only way to make logical sense of such an assertion is to interpret it as

Frege is intelligent, and Frege is ugly, or

$$\vdash \alpha \text{ and } \vdash \neg\beta.$$

Assumption V (Truth conditions of $\triangleright$)

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Indeed, anything weaker than that will not attain assertoric force: it will necessarily be <u>true to a non-maximal degree</u>, which is incompatible with Assumption I.

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To see this, assume that in this world, Frege is actually full-on intelligent, and somewhat ugly, though not a clear, indisputable case of ugliness.

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To see this, assume that in this world, Frege is actually full-on intelligent, and somewhat ugly, though not a clear, indisputable case of ugliness.

Then there is a possible (=logically consistent) world wherein Frege is full-on intelligent, and full-on ugly: total ugliness coupled with total intelligence is not an inconsistent prospect. In this possible world, then, $\alpha \rhd \beta$ must be true to a higher degree than it is in the world we initially considered, whence $\alpha \rhd \beta$ could not have been full-on true there.

Our last assumption subsumes Assumption V:

Assumption VI (Course of Values of $\rhd$)

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- The gap between the degree of truth of α and that of β; and
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This is the closest we get to the outright request that degrees of truth are magnitudes to be combined by arithmetic operations.

We are not quite asking that much, though. We are merely voicing the intuition that if, say, α is more true than β, then if the degree of truth of α grows while that of β stays constant, so does grow the degree of truth of $\alpha \rhd \beta$.

We can finally think clearly enough, and argue about, formulæ in the language

$$\neg, \rhd, \top.$$

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"Less true than"

$$\vdash \neg(\alpha \rhd \beta) \text{ if, and only if, } \alpha \text{ is no more true than } \beta.$$

For, if L.H.S. holds then it is full-on false that α is much more true than β. If we had "α more true than β" true to some degree, then we should have $\alpha \rhd \beta$ true, albeit possibly to a comparably small degree (Assumption VI). Hence "α no more true than β" holds.
Conversely, if "α no more true than β" holds, clearly $\alpha \rhd \beta$ is full-on false, hence $\vdash \neg(\alpha \rhd \beta)$.

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If, on the other hand, α is more true than β, then Assumption VI entails that the degree of truth of $(\star)$ is correlated, or "directly proportional", to that of β: hence it is reasonable, in this case, to claim that $(\star)$ agrees with β.

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Consider the formula:

$$\alpha \rhd (\alpha \rhd \beta). \qquad (\star)$$

If α is less true than β, $\alpha \rhd \beta$ is full-on false, and $\alpha \rhd \perp$ is just as true as α. Hence in this case $(\star)$ agrees with α.

If, on the other hand, α is more true than β, then Assumption VI entails that the degree of truth of $(\star)$ is correlated, or "directly proportional", to that of β: hence it is reasonable, in this case, to claim that $(\star)$ agrees with β.

Hence the degree of truth of $(\star)$ is the minimum of the degree of truths of α and β.

What about conjunctions and disjunctions?

Consider the formula:

$$\alpha \rhd (\alpha \rhd \beta). \qquad (\star)$$

If α is less true than β, $\alpha \rhd \beta$ is full-on false, and $\alpha \rhd \bot$ is just as true as α. Hence in this case $(\star)$ agrees with α.

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Hence the degree of truth of $(\star)$ is the minimum of the degree of truths of α and β.

Conjunction

We identify $\alpha \rhd (\alpha \rhd \beta)$ with the conjunction of α and β, written $\alpha \wedge \beta$. Observe: $\vdash \alpha \wedge \beta$ iff $\vdash \alpha$ and $\vdash \beta$.

Disjunction is defined through the De Morgan Laws.

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As an example of the foregoing:

Prelinearity

For any α and β we have:

$$\vdash \neg\,(\,(\alpha \rhd \beta) \wedge (\beta \rhd \alpha)\,)\,.$$

This is a version of the standard *prelinearity axiom* in many-valued logic: $\vdash (\alpha \to \beta) \vee (\beta \to \alpha)$.

In the present version, it states the obvious: it is always full-on false that α is much more true than β, and at the same time β is much more true than α.

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We now have a language, and an intended semantics. It's time to talk about inference.

How do we perform inference with vague propositions?

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Suppose $\alpha :=$"Ada is short", and $\beta :=$"Ada is fat". Suppose further:

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Under our assumptions, this is a perfectly valid inference. It is no less grounded than a classical inference. It is a form of *modus tollens*:

$$\frac{\vdash \alpha \rightarrow \beta \qquad\qquad \vdash \neg\beta}{\vdash \neg\alpha} \quad (\textsc{mt})$$

Only deduction rule we use: *vague modus tollens*.

$$\frac{\vdash \neg(\alpha \rhd \beta) \qquad\qquad \vdash \neg\beta}{\vdash \neg\alpha} \quad (\textsc{vmt})$$

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Now we declare that a formula α in the language $\{\neg, \rhd, \top\}$ is provable if there exists a proof of α, that is, a finite sequence of formulæ $\alpha_1, \ldots, \alpha_l$ a such that:

- $\alpha_l = \alpha$.
- Each α_i, $i < l$ is either an axiom, or is obtainable from α_j and α_k, $j, k < i$, via an application of *vague modus tollens*.

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So, what are the axioms?

$\boxed{\textit{Ex falso quodlibet}}$

$$\neg(\ \alpha \rhd \top\ )$$

$$\alpha \leqslant \top$$

Not much to say here: obvious.

$\boxed{\textit{A fortiori}}$

$$\alpha \rhd \beta \leqslant \alpha$$

For that by which α is truer than β cannot be less than the degree of truth of α itself. (In the extreme case, $\beta \equiv \perp$ and $\alpha \rhd \perp \equiv \alpha \leqslant \alpha$.)

$\boxed{\text{Transitivity of } \rhd}$

$$(\gamma \rhd \alpha) \rhd (\gamma \rhd \beta) \ \leqslant \ \beta \rhd \alpha$$

This is best understood through a lengthy case analysis (3 propositions). It is to be thought of a consequence essentially of our crucial Assumption VI about correlation: $\gamma \rhd \alpha$ and $\gamma \rhd \beta$ compare in respect of truth value in the opposite manner as α compares to β, hence the R.H.S. has them reversed. For example, if α is more true than β, then R.H.S. is full-on false, so L.H.S. should be, too. And indeed, assuming γ is more true than both α and β, that by which γ is truer than α is smaller than that by which γ is truer than β.

$\boxed{\text{Contraposition}}$

$$\alpha \rhd \beta \;\leqslant\; \neg\beta \rhd \neg\alpha$$

Not much to say here. By our interpretation of negation and symmetry, think equality in place of $\leqslant$.

$\boxed{\text{Conjunction is commutative}}$

$$\alpha \rhd (\alpha \rhd \beta) \leqslant \beta \rhd (\beta \rhd \alpha)$$

Once we accept that L.H.S. is $\alpha \wedge \beta$, and hence R.H.S. is $\beta \wedge \alpha$, not much to say here: conjunction is obviously commutative (again, think equality in place of $\leqslant$).

Axiom system.

(A0) $\neg(\alpha \rhd \top)$ *Ex falso quodlibet*

(A1) $\alpha \rhd \beta \leqslant \alpha$ *A fortiori*

(A2) $(\gamma \rhd \alpha) \rhd (\gamma \rhd \beta) \leqslant \beta \rhd \alpha$ Transitivity of $\rhd$

(A3) $\alpha \rhd (\alpha \rhd \beta) \leqslant \beta \rhd (\beta \rhd \alpha)$ Conjunction is commutative

(A4) $\alpha \rhd \beta \leqslant \neg\beta \rhd \neg\alpha$ Contraposition

$$\alpha \wedge \beta \equiv \alpha \rhd (\alpha \rhd \beta)$$

Deduction rule.

(R1) $\dfrac{\alpha \leqslant \beta \qquad \neg\beta}{\neg\alpha}$ *Vague Modus Tollens.*

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For example, our axiom defining conjunction standardly becomes

$$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$$

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For example, our axiom defining conjunction standardly becomes

$$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$$

and now states that *disjunction* is commutative.

We have:

$$\alpha \rhd \beta \equiv \neg(\alpha \to \beta).$$

The connective that I am denoting $\rhd$ is usually denoted $\ominus$.

Numbers out of Formulæ

Otto Hölder, 1859–1937.

Consider the vague proposition,

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In Łukasiewicz logic:

The degree of truth attached to X (in a given possible world, i.e. valuation) is the set of answers to a tree of yes/no questions.

$$\boxed{\vdash \alpha\ ?}$$

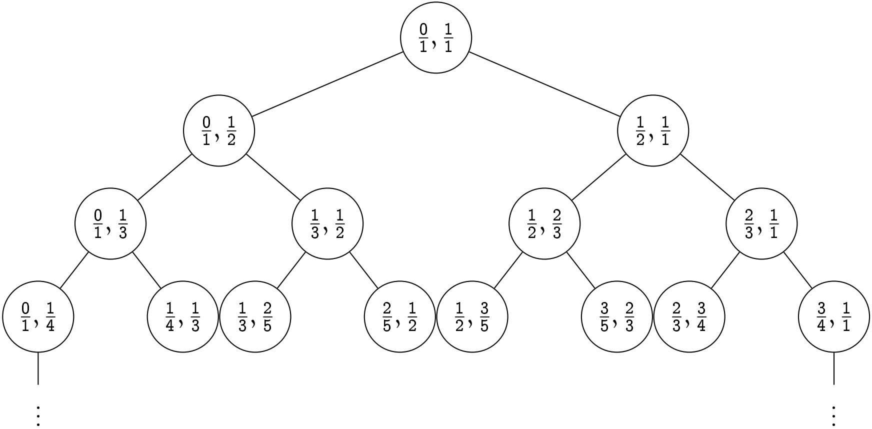
$$\begin{aligned} L_{0,1} &= \neg X \\ R_{0,1} &= X \end{aligned}$$

$$\begin{aligned} L_{1,1} &= L_{0,1} \rhd R_{0,1} \\ R_{1,1} &= R_{0,1} \end{aligned} \qquad \begin{aligned} L_{1,2} &= L_{0,1} \\ R_{1,2} &= R_{0,1} \rhd L_{0,1} \end{aligned}$$

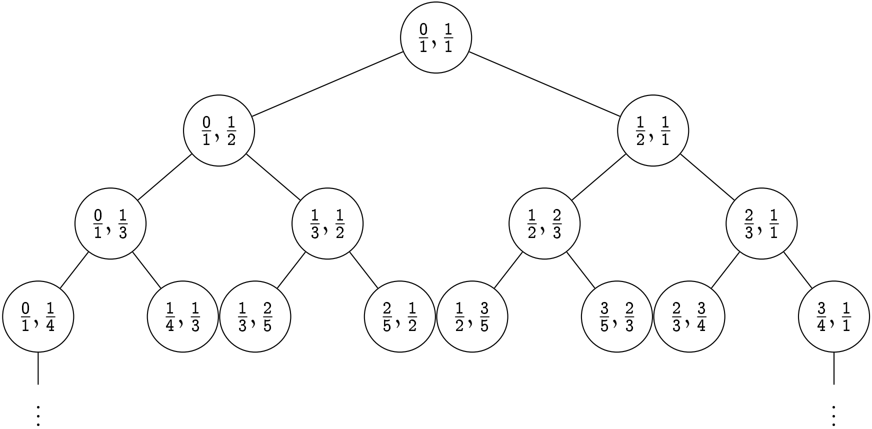
$$\begin{aligned} L_{2,1} &= L_{1,1} \rhd R_{1,1} \\ R_{2,1} &= R_{1,1} \end{aligned} \quad \begin{aligned} L_{2,2} &= L_{1,1} \\ R_{2,2} &= R_{1,1} \rhd L_{1,1} \end{aligned} \quad \begin{aligned} L_{2,3} &= L_{1,2} \rhd R_{1,2} \\ R_{2,3} &= R_{1,2} \end{aligned} \quad \begin{aligned} L_{2,4} &= L_{1,2} \\ R_{2,4} &= R_{1,2} \rhd L_{1,2} \end{aligned}$$

$$\begin{matrix} L_{3,1} & L_{3,2} & L_{3,3} & L_{3,4} & L_{3,5} & L_{3,6} & L_{3,7} & L_{3,8} \\ R_{3,1} & R_{3,2} & R_{3,3} & R_{3,4} & R_{3,5} & R_{3,6} & R_{3,7} & R_{3,8} \\ \vdots & & & & & & & \vdots \end{matrix}$$

The Yes/No Questions.



The Farey tree.



Cauchy's Theorem. Every rational number in $(0, 1)$ occurs, automatically in reduced form, as the mediant of the numbers in some node of the Farey tree exactly once. (The mediant of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a+c}{b+d}$.)

$$\begin{aligned} L_{0,1} &= \neg X \\ R_{0,1} &= X \end{aligned}$$

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Thm. There are natural bijections between the finitely axiomatisable maximal consistent theories in Ł over 1 variable X, the nodes of the Farey tree together with $\{0, 1\}$, and the rational numbers in $[0, 1]$.

This correspondence can be extended to a natural correspondence with all numbers in $[0, 1]$, removing the assumption of finite axiomatisability. (There is a further and final extension to *numbers in* $[0, 1]$ *plus a linear infinitesimal*, by considering all *prime* theories, but I will not discuss it here.)

Something to take home.

The innocent-looking Łukasiewicz axioms (A0–A4) determine the real numbers.

Carolina, born 2009.

Thank you for your attention.