On (Uniform) Interpolation in Non-Classical Logics

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SGSLPS Workshop on Many-Valued Logics

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Interpolation in classical FO logic

Theorem ("Lemma 3" in Craig, 1957)

Let φ, ψ be sentences of first-order logic such that $\vdash \varphi \to \psi$.

$$\varphi \quad \to \quad \quad \to \quad \psi$$

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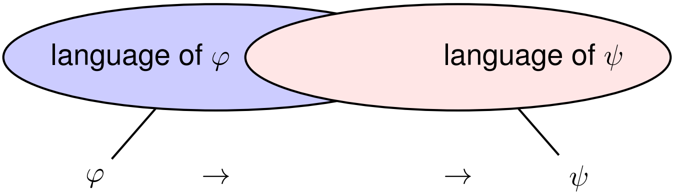

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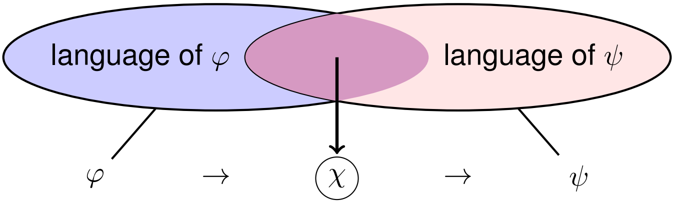
Let φ, ψ be sentences of first-order logic such that $\vdash \varphi \to \psi$. There exists a sentence χ such that

- $\mathrm{Rel}(\chi) \subseteq \mathrm{Rel}(\varphi) \cap \mathrm{Rel}(\psi)$,
- $\vdash \varphi \to \chi$, *and*
- $\vdash \chi \to \psi$.



Origins

"Although I was aware of the mathematical interest of questions related to elimination problems in logic, my main aim, initially unfocused, was to try to use methods and results from logic to clarify or illuminate a topic that seems central to empiricist programs: In epistemology, the relationship between the external world and sense data; in philosophy of science, that between theoretical constructs and observed data."

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Applications to mathematical logic:

- Separating projective classes by an elementary class;
- (Beth 1953) Implicit definability implies explicit definability.

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- An algebraic viewpoint on interpolation;
- The more general property of uniform interpolation.

Warm-up: Classical Propositional Logic

Theorem (Interpolation in Classical Propositional Logic)

Let $\varphi(\overline{p}, \overline{q})$ *and* $\psi(\overline{p}, \overline{r})$ *be two propositional formulas such that* $\vdash_{\mathrm{CPC}} \varphi \to \psi$*. There exists a propositional formula* $\chi(\overline{p})$ *such that* $\vdash_{\mathrm{CPC}} \varphi \to \chi$ *and* $\vdash_{\mathrm{CPC}} \chi \to \psi$*.*

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$\varphi : \neg(q \to p)$, $\psi : p \to \neg r$

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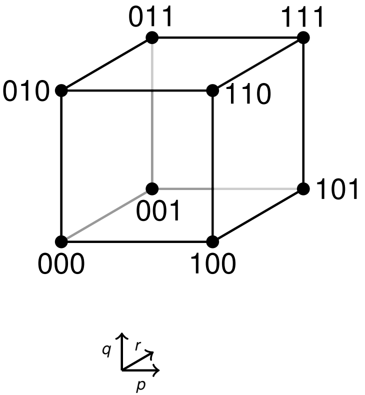

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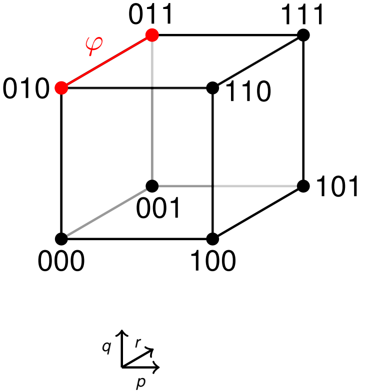

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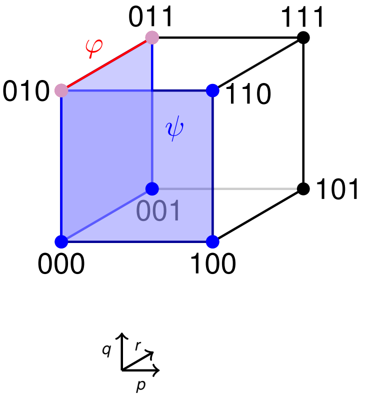

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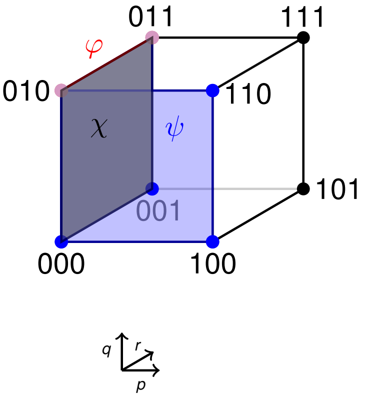

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Note: the formula $\chi(\overline{p})$ does not depend on ψ! It is also denoted $\exists_{\overline{q}}\varphi$ and is a uniform interpolant for φ; see later in this talk.

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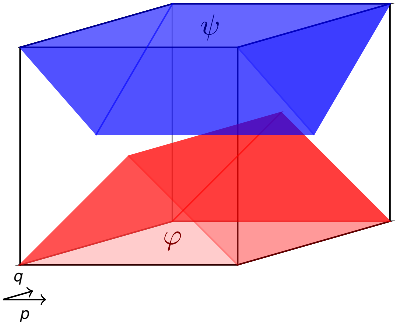

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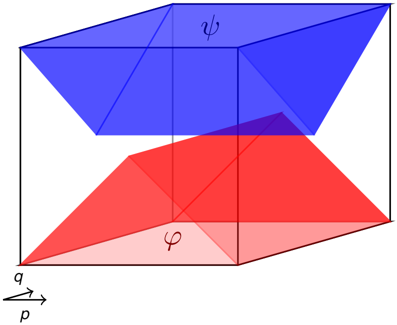
Then $\vdash_{Ł} \varphi \to \psi$, but there is no formula χ without variables such that $\vdash_{Ł} \varphi \to \chi$ and $\vdash_{Ł} \chi \to \psi$.



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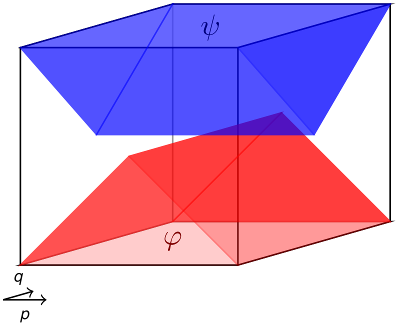

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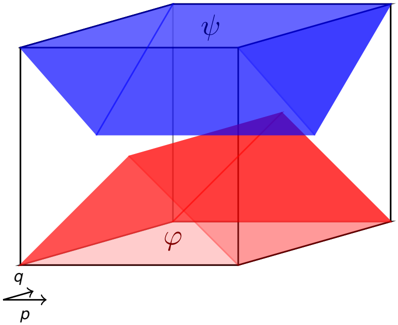

This failure of Craig interpolation is closely related to the failure of the deduction theorem: $\varphi \vdash_Ł 0$, but $\nvdash_Ł \varphi \to 0$.

Deductive Interpolation in Ł

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Let $\varphi(\overline{p}, \overline{q})$ and $\psi(\overline{p}, \overline{r})$ be formulas of Ł. If $\varphi \vdash_{Ł} \psi$, then there exists a formula $\chi(\overline{p})$ of Ł such that $\varphi \vdash_{Ł} \chi$ and $\chi \vdash_{Ł} \psi$.

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The fact that χ is indeed an interpolant is most easily seen in a picture ...

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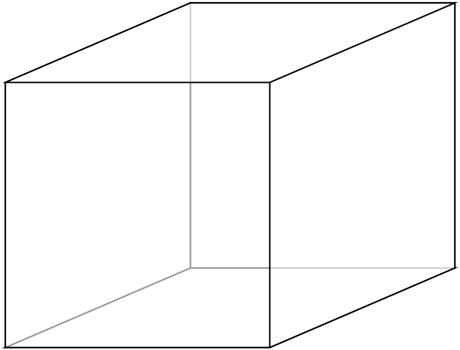

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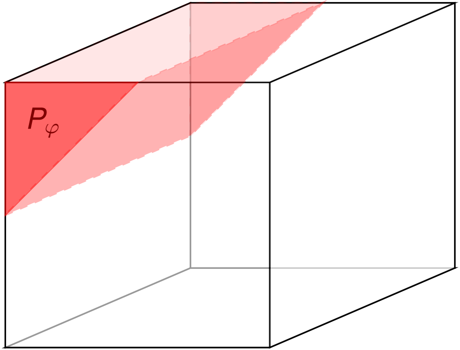

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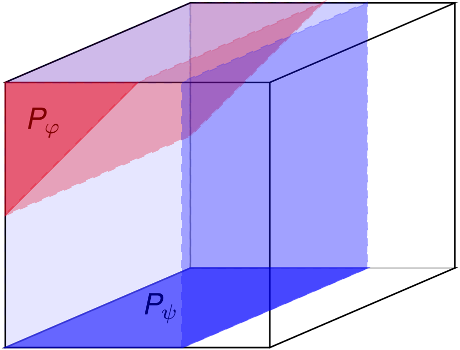

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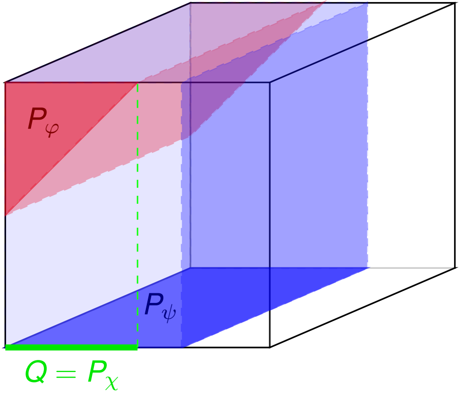

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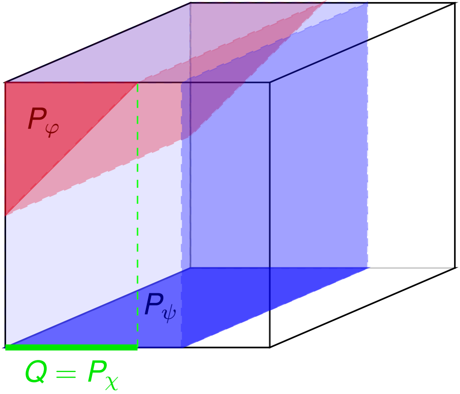

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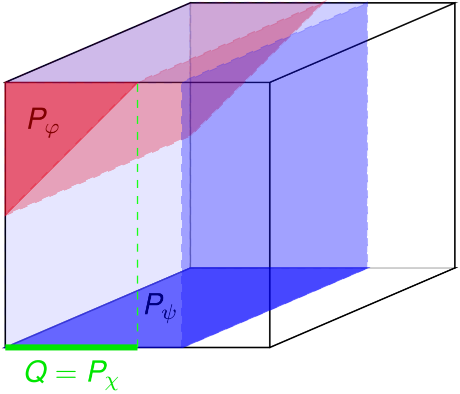

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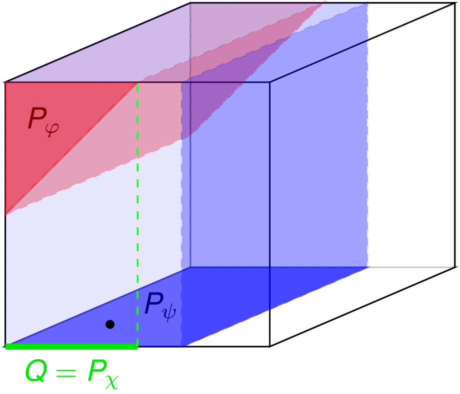

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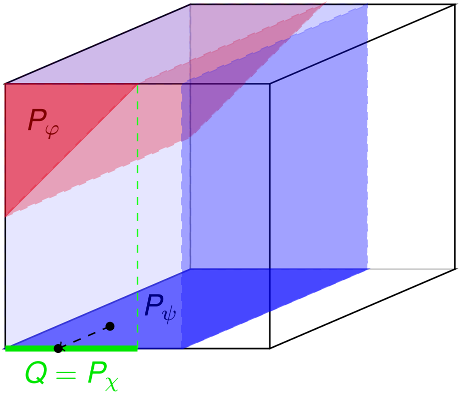

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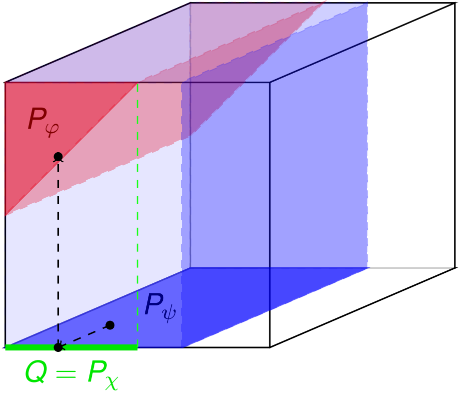

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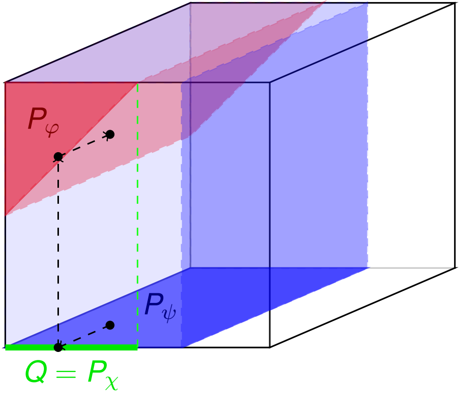

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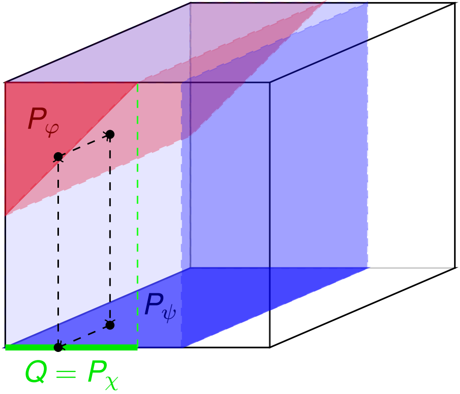

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Deductive Interpolation, algebraic view

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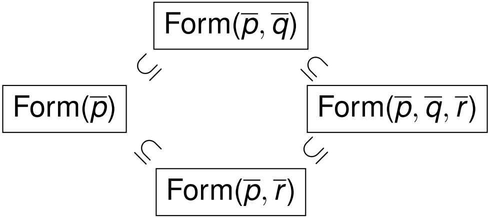
- We now make a brief excursion into the general algebraic phenomena related to Deductive Interpolation.
- This will be useful for proving Deductive Interpolation for Gödel-Dummett logic.

Deductive Interpolation, algebraic view



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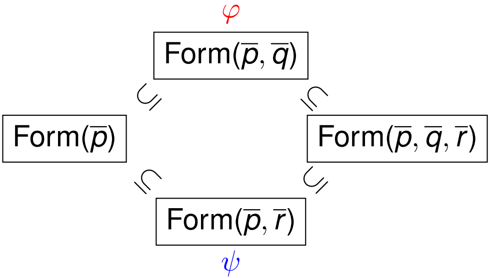

Deductive Interpolation, algebraic view

φ

$\mathrm{Form}(\overline{p}) \subseteq \mathrm{Form}(\overline{p}, \overline{q}) \subseteq \mathrm{Form}(\overline{p}, \overline{q}, \overline{r})$

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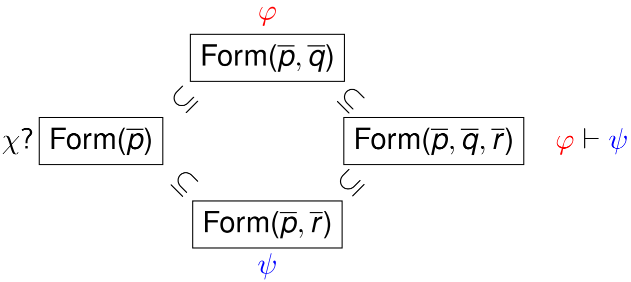
Deductive Interpolation, algebraic view



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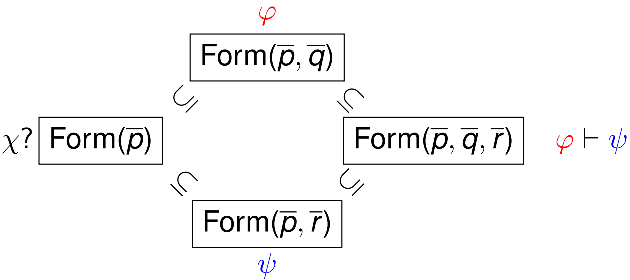

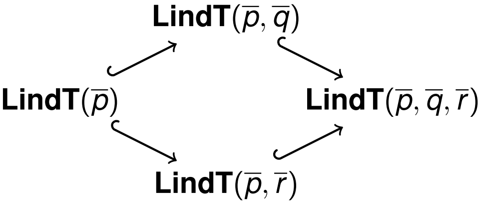

Deductive Interpolation, algebraic view

$$
\begin{array}{ccccc}
 & & \varphi & & \\
 & & \boxed{\mathsf{Form}(\overline{p},\overline{q})} & & \\
 & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\supseteq$} & \\
\chi?\ \boxed{\mathsf{Form}(\overline{p})} & & & & \boxed{\mathsf{Form}(\overline{p},\overline{q},\overline{r})} \qquad \varphi \vdash \psi \\
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 & & \psi & &
\end{array}
$$

$$
\begin{array}{ccccc}
 & & [\varphi] & & \\
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 & \nearrow & & \searrow & \\
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- Equational consequence ($\Phi \models_{\mathcal{V}} \psi$) coincides with logical consequence ($\Phi \vdash_{\mathbf{L}} \psi$).

Deductive Interpolation, algebraic view

Definition

A class of algebras $\mathcal{V}$ has deductive interpolation if, for every set of equations $\Phi(\overline{p}, \overline{q})$ and an equation $\psi(\overline{p}, \overline{r})$ such that $\Phi \models_{\mathcal{V}} \psi$, there exists a set of equations $\Pi(\overline{p})$ such that $\Phi \models_{\mathcal{V}} \Pi$ and $\Pi \models_{\mathcal{V}} \psi$.

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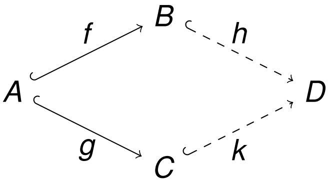

Interpolation and amalgamation

Theorem

Let $\mathcal{V}$ be a variety. Consider the properties:

1. *$\mathcal{V}$ has deductive interpolation,*
2. *For any finite $\overline{p}$, $\overline{q}$, $\overline{r}$, and θ a congruence on $\mathbf{F}_{\mathcal{V}}(\overline{p}, \overline{q})$,*

$$\langle\theta\rangle_{\mathbf{F}_{\mathcal{V}}(\overline{p},\overline{q},\overline{r})} \cap \mathbf{F}_{\mathcal{V}}(\overline{p}, \overline{r}) = \langle\theta \cap \mathbf{F}_{\mathcal{V}}(\overline{p})\rangle_{\mathbf{F}_{\mathcal{V}}(\overline{p},\overline{r})}.$$

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For any variety $\mathcal{V}$, we have (1) $\Leftrightarrow$ (2) $\Leftarrow$ (3).
If, moreover, $\mathcal{V}$ has the congruence extension property, then all three properties are equivalent.

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(Fact. MV-algebras and Gödel algebras have the CEP.)

If you thought that was complicated...

$$
\begin{array}{ccccccc}
\text{CEP} + \text{FAP} & & & & & & \\
\Updownarrow & & & & & & \\
\text{TIP} & \Longrightarrow & \text{AP} & \Longrightarrow & \text{WAP} & \Longrightarrow & \text{FAP} \\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow \\
\text{MIP} & \Longrightarrow & \text{RP} & \Longrightarrow & \text{CDIP} & \Longrightarrow & \text{DIP} \\
\Updownarrow & & & & & & \\
\text{DIP} + \text{EP} & & & & & &
\end{array}
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Metcalfe, Montagna, Tsinakis (2014)

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The variety of Gödel algebras has amalgamation.

Proof by Picture.

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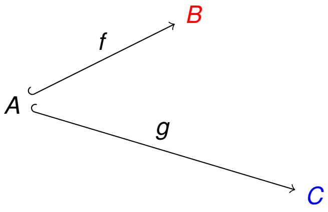

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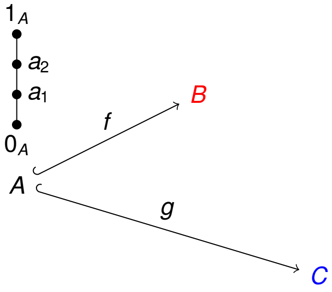

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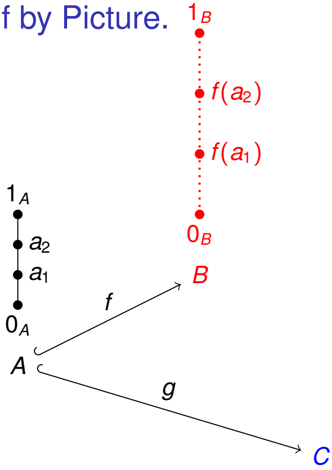

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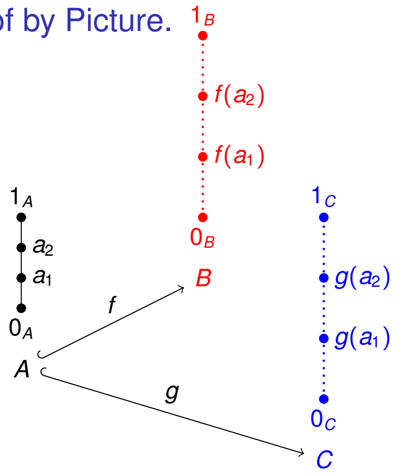

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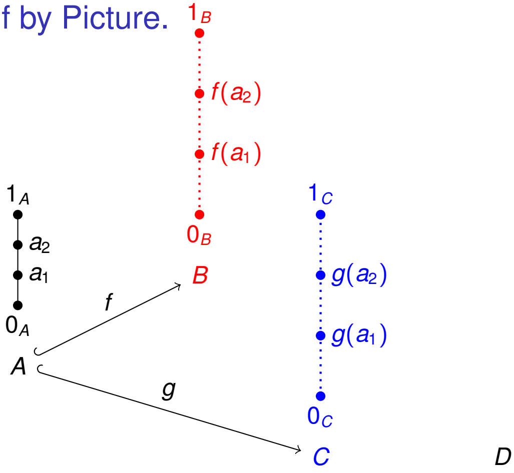

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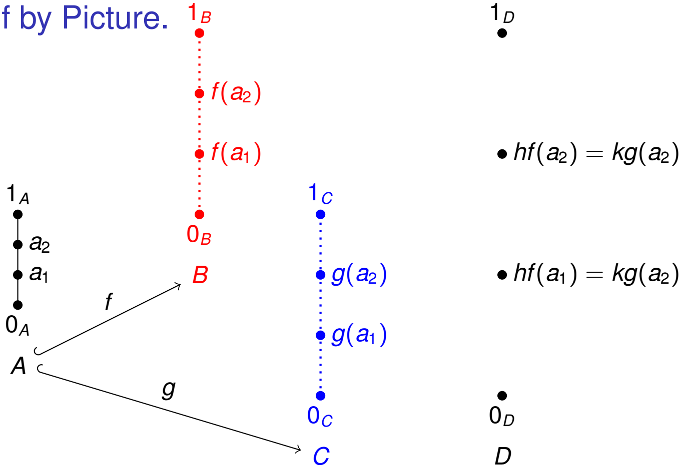

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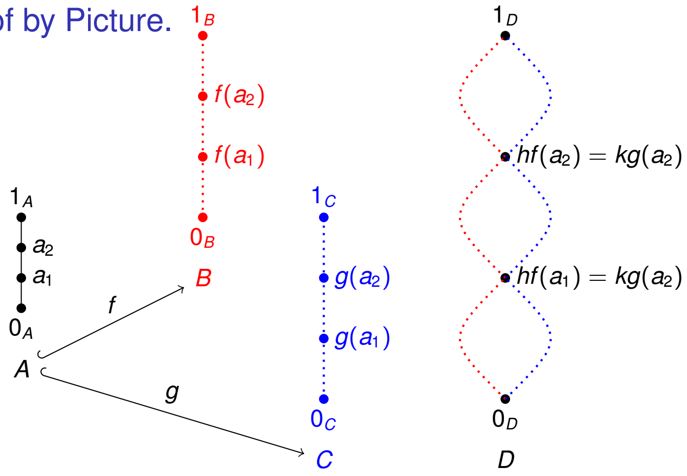

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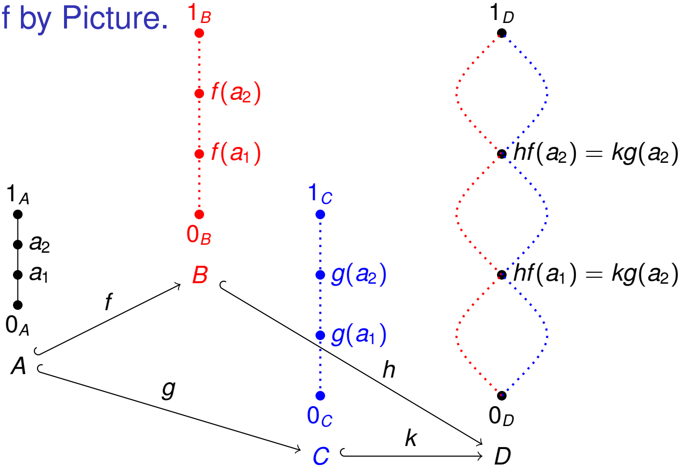

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Amalgamation of Gödel algebras

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Proof.

It suffices to prove it for Gödel chains (Lemma).
Let $f : A \hookrightarrow B$ and $g : A \hookrightarrow C$ be injective homomorphisms.
Define the set $D := (B \sqcup C)/\sim$, where $\sim$ identifies $f(a)$ and $g(a)$ for every $a \in A$.
Write $d_1 \preceq_D d_2$ just in case one of the following holds:

- $d_1, d_2 \in B$ and $d_1 \leq_B d_2$;
- $d_1, d_2 \in C$ and $d_1 \leq_C d_2$;
- $d_1 \in B$, $d_2 \in C$, $d_1 \leq_B f(a)$ and $g(a) \leq_C d_2$ for some $a \in A$;
- $d_1 \in C$, $d_2 \in B$, $d_1 \leq_C g(a)$ and $f(a) \leq_B d_2$ for some $a \in A$.

Then $\preceq_D$ is a partial order on D, and any extension of $\preceq_D$ to a total order $\leq_D$ yields an amalgamating Gödel chain. □

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Gödel-Dummett Logic has Deductive Interpolation.

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Maksimova (1977) proved that there are exactly 8 logics between intuitionstic and classical propositional logic that have interpolation. (There are continuum many logics between IPC and CPC!)

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The (usual) interpolation property ensures that $\exists_{\overline{q}}\varphi$ is a uniform interpolant. The definition of $\forall_{\overline{q}}\varphi$ is similar (exercise). □

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- Morally, having uniform interpolation means having an ‘internal representation’ of second-order quantification inside the logic.
- Also see: several papers by Ghilardi and Zawadowski, and my paper joint with Metcalfe and Tsinakis at TACL 2015.

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Conclusion

- Both Łukasiewicz and Gödel-Dummett logic enjoy interpolation properties;
- Algebraic and semantic methods are useful for proving this;
- At the first-order level, many problems are open, notably: does the predicate version of Gödel-Dummett logic have interpolation?
- Just as ‘normal’ interpolation, uniform interpolation also corresponds to beautiful properties of the associated class of algebras; notably with the ‘existentially closed’ algebras. This deserves more investigation.

On (Uniform) Interpolation in Non-Classical Logics

Sam van Gool

Dipartimento di Matematica "Federigo Enriques"
Università degli Studi di Milano

SGSLPS Workshop on Many-Valued Logics

22 May 2015, Bern