Geometric aspects of Łukasiewicz logic
A short excursion

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The basics of Łukasiewicz logic

Jan Łukasiewicz, 1878–1956.

We start with a (finite or infinite) set of propositional variables, or atomic formulæ, that are to stand for propositions. Say, if we content ourselves with countably many:

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To construct compound formulæ we use the logical connectives:

- $\vee$, for disjunction ("inclusive or", Latin *vel*);
- $\wedge$, for conjunction ("and", Latin *et*);
- $\rightarrow$, for implication ("if. . . then. . . ", conditional assertions);
- $\neg$, for negation ("not", negative assertions).

The usual recursive definition of general formulæ now reads as follows.

- $\top$ and $\bot$ are formulæ.
- All propositional variables are formulæ.
- If α and β are formulæ, so are $(\alpha \vee \beta)$, $(\alpha \wedge \beta)$, $(\alpha \rightarrow \beta)$, and $\neg\alpha$.
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- Nothing else is a formula.

Let us write FORM for the set of all formulæ constructed over the countable language $X_1, \ldots, X_n, \ldots$. Observe that formulæ are defined exactly in the same manner in classical logic.

We now define a formal semantics for our logic. Łukasiewicz logic has a many-valued semantics: specifically, we take $[0, 1] \subseteq \mathbb{R}$ as a set of "truth values".

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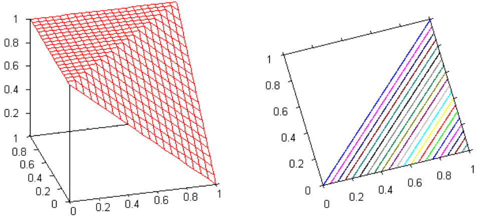
- $w(\neg\alpha) = 1 - w(\alpha)$.
- $w(\alpha \to \beta) = \begin{cases} 1 & \text{if } w(\alpha) \leqslant w(\beta) \\ 1 - (w(\alpha) - w(\beta)) & \text{otherwise.} \end{cases}$



Truth-function of Łukasiewicz implication.

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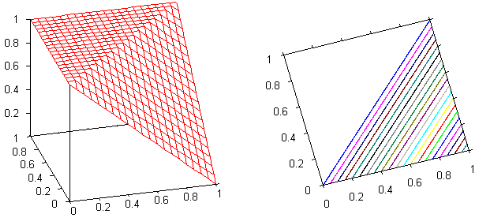

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$$w(\alpha \to \beta) = \min\{1, 1 - (w(\alpha) - w(\beta))\}$$

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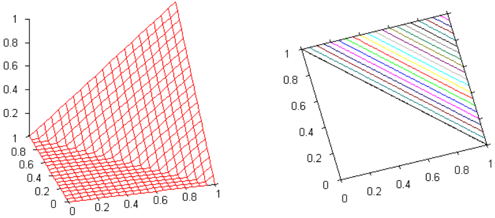
Notation	Definition	Name
$\bot$	–	*Falsum*
$\top$	$\neg\bot$	*Verum*
$\neg\alpha$	–	Negation
$\alpha \rightarrow \beta$	–	Implication
$\alpha \vee \beta$	$(\alpha \rightarrow \beta) \rightarrow \beta$	(Lattice) Disjunction
$\alpha \wedge \beta$	$\neg(\neg\alpha \vee \neg\beta)$	(Lattice) Conjunction
$\alpha \leftrightarrow \beta$	$(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$	Biconditional
$\alpha \oplus \beta$	$\neg\alpha \rightarrow \beta$	Strong disjunction
$\alpha \odot \beta$	$\neg(\alpha \rightarrow \neg\beta)$	Strong conjunction
$\alpha \ominus \beta$	$\neg(\alpha \rightarrow \beta)$	Co-implication

Table: Connectives in Łukasiewicz logic.

The corresponding formal semantics is as follows:

Notation	Formal semantics
$\bot$	$w(\bot) = 0$
$\top$	$w(\top) = 1$
$\neg\alpha$	$w(\neg\alpha) = 1 - w(\alpha)$
$\alpha \to \beta$	$w(\alpha \to \beta) = \min\{1, 1 - (w(\alpha) - w(\beta))\}$
$\alpha \vee \beta$	$w(\alpha \vee \beta) = \max\{w(\alpha), w(\beta)\}$
$\alpha \wedge \beta$	$w(\alpha \wedge \beta) = \min\{w(\alpha), w(\beta)\}$
$\alpha \leftrightarrow \beta$	$w(\alpha \leftrightarrow \beta) = 1 - \lvert w(\alpha) - w(\beta)\rvert$
$\alpha \oplus \beta$	$w(\alpha \oplus \beta) = \min\{1, w(\alpha) + w(\beta)\}$
$\alpha \odot \beta$	$w(\alpha \odot \beta) = \max\{0, w(\alpha) + w(\beta) - 1\}$
$\alpha \ominus \beta$	$w(\alpha \ominus \beta) = \max\{0, w(\alpha) - w(\beta)\}$

Table: Formal semantics of connectives in Łukasiewicz logic.



Truth-function of Łukasiewicz "strong conjunction" $\odot$*.*
(Note: Non-idempotent operation.)

$$w(\alpha \odot \beta) = \max\{0, w(\alpha) + w(\beta) - 1\}$$

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- $\bot \to \alpha$ (*Ex falso quodlibet*)
- $\sout{\alpha \vee \neg\alpha}$ (*Tertium non datur*)
- $\sout{\neg(\alpha \wedge \neg\alpha)}$ (Principle of non-contradiction)
- $\neg\neg\alpha \to \alpha$ (Law of double negation)
- $\sout{(\neg\alpha \to \alpha) \to \alpha}$ (*Consequentia mirabilis*)
- $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$ (Contraposition)
- $(\alpha \to \beta) \vee (\beta \to \alpha)$ (Pre-linearity)

Define: $\textsc{Taut} \subseteq \textsc{Form}$ is the set of all tautologies. Write: $\vDash \alpha$ to mean $\alpha \in \textsc{Taut}$.

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To define provability, we select (with a lot of hindsight) a set of tautologies, and declare that they are axioms: they count as provable formulæ by definition.

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Next we select a set of deduction rules that tell us that if we already established that formulæ $\alpha_1, \ldots, \alpha_n$ are provable, and these have a certain shape, then a specific formula β is also a provable formula.

Most important deduction rule (only one we use): modus ponens.

$$\frac{\alpha \qquad \alpha \to \beta}{\beta} \quad (\text{MP})$$

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Now we declare that a formula $\alpha \in \textsc{Form}$ is provable if there exists a proof of α, that is, a finite sequence of formulæ $\alpha_1, \ldots, \alpha_l$ a such that:

- $\alpha_l = \alpha$.
- Each α_i, $i < l$ is either an axiom, or is obtainable from α_j and α_k, $j, k < i$, via an application of *modus ponens*.

Define: $\textsc{Thm} \subseteq \textsc{Form}$ is the set of provable formulæ. Write: $\vdash \alpha$ to mean $\alpha \in \textsc{Thm}$.

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We still need to define the axioms.

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Upon defining

$$\alpha \vee \beta \equiv (\alpha \to \beta) \to \beta$$

(A0–A5) read as shown next.

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Deduction rule for classical logic.

(R1) $\dfrac{\alpha \qquad \alpha\to\beta}{\beta}$ *Modus ponens.*

Axiom system for Łukasiewicz logic.

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Deduction rule for Łukasiewicz logic.

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Such “succint descriptions” can be polysemous to a surprising extent indeed.

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Moral: The import of removing one axiom from an axiom system depends on the axiom system itself. In particular, Hilbert-style systems are of little use to analyse the structural properties of logics in terms of a specific axiomatisation.

(For this, the Gentzen-style systems used in proof theory are more useful.)

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This concludes our definition of Łukasiewicz (propositional) logic.

A first important result. In Łukasiewicz logic, the relationship between tautologies and theorems is entirely analogous to the one that holds in classical logic. It is stated in the next result, a substantial piece of mathematics:

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Soundness and Completeness Theorem for Ł

$$\textsc{Taut} = \textsc{Thm}\,.$$

A. Rose and J. Barkley Rosser, *Trans. of the AMS*, 1958.

Proof is syntactic. Algebraic proof given shortly thereafter by C.C. Chang, which introduced MV-algebras for this purpose. We will return to them if time allows.

Classical logic satisfies a stronger completeness theorem. For $S, \{\alpha\} \subseteq \textsc{Form}$, write $S \vdash \alpha$ if α is provable form the logical axioms augmented by S, and $S \models \alpha$ if α holds in each model (=possible world, assignment) wherein each formula of S holds.

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Strong Completeness Theorem for CL

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In the actual use of any logic, it is of great importance to have completeness under additional sets S of assumptions. It is S that encodes our knowledge about a specific application domain. Pure logic ($S = \emptyset$) can teach us nothing about the world, by definition.

Łukasiewicz logic fails strong completeness.

Let S be the set of formulæ in one variable p:

$$\varphi_n(p) := ((n+1)(p^n \wedge \neg p)) \oplus p^{n+1} \,,$$

for each integer $n \geqslant 1$, where

$$p^k := \underbrace{p \odot \cdots \odot p}_{k \text{ times}} \,,$$

$$kp := \underbrace{p \oplus \cdots \oplus p}_{k \text{ times}} \,.$$

Then $S \not\vdash_{Ł} p$, but $S \vDash_{Ł} p$.

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Intuitively, you can think of S as embodying the following infinite set of assumptions:

1. p :="Enzo is tall" is true to degree $\geqslant 1/2$.
2. p :="Enzo is tall" is true to degree $\geqslant 2/3$.
3. p :="Enzo is tall" is true to degree $\geqslant 3/4$.
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Taking stock. $\vdash_{Ł}$ is **compact**, but $\vDash_{Ł}$ is not.

Note. $S \vdash_{Ł} \alpha \Rightarrow S \vDash_{Ł} \alpha$ always.

The Hay-Wójcicki Theorem:

Completeness Theorem for f.a. theories in Ł

For any $\alpha \in \textsc{Form}$, and any finite set $F \subseteq \textsc{Form}$,

$$F \vDash_{\text{Ł}} \alpha \quad \text{if, and only if,} \quad F \vdash_{\text{Ł}} \alpha\,.$$

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A folklore theorem:

Completeness Theorem for maximal theories in Ł

For any $\alpha \in \textsc{Form}$, and any maximal consistent set $M \subseteq \textsc{Form}$,

$$M \vDash_{Ł} \alpha \quad \text{if, and only if,} \quad M \vdash_{Ł} \alpha\,.$$

Satisfiability and consistency in Ł

Notion	Definition	Description
α is satisfiable	$\exists w$ such that $w(\alpha) = 1$	α is 1-satisfiabile
α is consistent	$\exists \beta$ such that $\alpha \nvdash_{Ł} \beta$	α does not prove smthg.
α is unsatisfiable	$\forall w$ we have $w(\alpha) < 1$	α is not 1-satisfiable
α is inconsistent	$\forall \beta$ we have $\alpha \vdash_{Ł} \beta$	α proves everything
α is strongly unsat.	$\forall w$ we have $w(\alpha) = 0$	α is always false
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Nota Bene. The terminology "Strongly unsatisfiable/inconsistent" is not standard. I only use it for ease of exposition. I do not know of a standard terminology for these concepts.

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Equivalent in classical logic by the Principle of Bivalence.

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Notion	Definition	Description
α is satisfiable	$\exists w$ such that $w(\alpha) = 1$	α is 1-satisfiabile
α is consistent	$\exists \beta$ such that $\alpha \nvdash_{Ł} \beta$	α does not prove smthg.
α is unsatisfiable	$\forall w$ we have $w(\alpha) < 1$	α is not 1-satisfiable
α is inconsistent	$\forall \beta$ we have $\alpha \vdash_{Ł} \beta$	α proves everything
α is strongly unsat.	$\forall w$ we have $w(\alpha) = 0$	α is always false
α is strongly incon.	$\forall \beta$ we have $\vdash_{Ł} \alpha \to \beta$	α implies everything

Equivalent in classical logic by the Deduction Theorem.

Deduction Theorem for CL

For any $\alpha, \beta \in \textsc{Form}$,

$$\alpha \vdash \beta \quad \text{if, and only if,} \quad \vdash \alpha \rightarrow \beta\,.$$

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Local Deduction Theorem for Ł

For any $\alpha, \beta \in \textsc{Form}$,

$$\alpha \vdash_{Ł} \beta \quad \text{if, and only if,} \quad \exists n \geqslant 1 \text{ such that } \vdash_{Ł} \alpha^n \rightarrow \beta\,.$$

(Notation: $\alpha^n := \underbrace{\alpha \odot \cdots \odot \alpha}_{n \text{ times}}$.)

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1. We cannot think of $\alpha \to \beta$ as "from the assumption of α, there follows β", i.e. as $\alpha \vdash \beta$. The Łukasiewicz implication is not a conditional.
2. The (Fregean) distinction between asserting a proposition and contemplating that proposition becomes essential: in classical logic, deduction theorem+bivalence make that distinction far less important. (Cf. the Tarskian identification of the meaning of a proposition α with its truth conditions: this fails badly in Łukasiewicz logic.)

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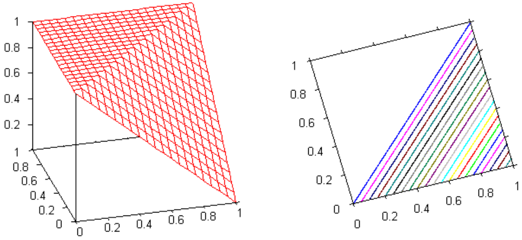
3. As a consequence of the two previous items, while it is easy to say what the assertion $\vdash \alpha \to \beta$ means, it is far harder to say what the plain proposition $\alpha \to \beta$ means. In other words, the intended meaning of the connective $\to$ is unclear.

Symbol	Name	Classically read
$\top$	*verum*	Always true
$\bot$	*falsum*	Always false
$\vee$	disjunction	Inclusive or (*vel*)
$\wedge$	conjunction	And
$\to$	implication	If...then...
$\neg$	negation	Not

Notation	Definition	Formal Semantics
$\top$	$\neg\bot$	$w(\top) = 1$
$\alpha \vee \beta$	$(\alpha \to \beta) \to \beta$	$w(\alpha \vee \beta) = \max\{w(\alpha), w(\beta)\}$
$\alpha \wedge \beta$	$\neg(\neg\alpha \vee \neg\beta)$	$w(\alpha \wedge \beta) = \min\{w(\alpha), w(\beta)\}$
$\alpha \leftrightarrow \beta$	$(\alpha \to \beta) \wedge (\beta \to \alpha)$	$w(\alpha \leftrightarrow \beta) = 1 - \lvert w(\alpha) - w(\beta)\rvert$
$\alpha \oplus \beta$	$\neg\alpha \to \beta$	$w(\alpha \oplus \beta) = \min\{w(\alpha) + w(\beta), 1\}$
$\alpha \ominus \beta$	$\neg(\alpha \to \beta)$	$w(\alpha \ominus \beta) = \max\{w(\alpha) - w(\beta), 0\}$

Table: Connectives in Łukasiewicz logic.



Truth-function of Łukasiewicz implication.

$$w(\alpha \to \beta) = \min\{1, 1 - (w(\alpha) - w(\beta))\}$$

$$w(\alpha \to \beta) = \begin{cases} 1 & \text{if } w(\alpha) \leqslant w(\beta) \\ 1 - (w(\alpha) - w(\beta)) & \text{otherwise.} \end{cases}$$

MV-algebras

C. C. Chang in Rome, 1969.

MV-algebras

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Lindenbaum's Equivalence Relation

Say $\alpha, \beta \in \textsc{Form}$ are logically equivalent if $\vdash \alpha \leftrightarrow \beta$. Write $\alpha \equiv \beta$.

On the quotient set $\frac{\textsc{Form}}{\equiv}$, the connectives induce operations:

- $0 := [\bot]_{\equiv}$
- $\neg[\alpha]_{\equiv} := [\neg\alpha]_{\equiv}$
- $[\alpha]_{\equiv} \oplus [\beta]_{\equiv} := [\alpha \oplus \beta]_{\equiv}$

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MV-algebras : Łukasiewicz logic = Boolean algebras : Classical logic

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$$\frac{\text{MV-algebras : Łukasiewicz logic} = \text{Boolean algebras : Classical logic}}{}$$

Abstractly: $(M, \oplus, \neg, 0)$ is an MV-algebra if $(M, \oplus, 0)$ is a commutative monoid, $\neg\neg x = x$, $1 := \neg 0$ is absorbing for $\oplus$ $(x \oplus 1 = 1)$, and, characteristically,

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x \qquad (*)$$

Any MV-algebra has an underlying distributive lattice bounded below by 0 and above by 1. Joins are given by

$$x \vee y := \neg(\neg x \oplus y) \oplus y$$

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Boolean algebras=Idempotent MV-algebras: $x \oplus x = x$.
Equivalently: MV-algebras that satisfy the *tertium non datur* law

$$x \vee \neg x = 1$$

.

The interval $[0,1] \subseteq \mathbb{R}$ can be made into an MV-algebra with neutral element 0 by defining

$$x \oplus y := \min\{x+y, 1\} \quad , \quad \neg x := 1 - x \, .$$

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Theorem (Chang's completeness theorem, 1959)

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This means: The class of MV-algebras coincides with $\mathrm{HSP}([0,1])$ — any MV-algebra can be represented as a homomorphic image of a subalgebra of a product of copies of $[0,1]$.

Or: The equations (in the language of MV-algebras) that hold in all MV-algebras are exactly those that hold in $[0,1]$.

Or: Any $\alpha \in \textsc{Form}$ that has a counter-model in some MV-algebra, already has a counter-model in $[0,1]$.

Let us consider the *tertium non datur* equation:

$$x \vee \neg x \;=\; 1\,. \tag{$\star$}$$

Then $(\star)$ is not an identity over $[0,1]$: the only evaluations into $[0,1]$ that satisfy $(\star)$ are $x \mapsto 0$ and $x \mapsto 1$ — the Boolean, or classical, evaluations.

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The boundary of the unit square.

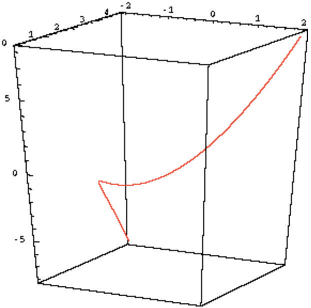
$$X \vee \neg X = 1 \qquad (\star)$$

The boundary of the unit interval.

$$X \vee \neg X \vee Y \vee \neg Y = 1 \qquad (\star\star)$$

The boundary of the unit square.

The *twisted cubic*: $\mathbb{V}\left(\{y - x^2, z - x^3\}\right)$



(Parametrisation: $t \longmapsto (t, t^2, t^3)$.)

Rational polyhedra

Leonardo's Truncated Icosahedron

(*Illustration for Luca Pacioli's The Divine Proportion, 1509.*)

We consider finitely presented MV-algebras, *i.e.* those of the form $\mathscr{F}_n/\theta$, with θ a finitely generated congruence (ideal). The assumption on θ is far from immaterial: there is no Hilbert's Basis Theorem for MV-algebras.

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The convex hull of a set $P \subseteq \mathbb{R}^n$, written $\operatorname{conv} P$, is the collection of all convex combinations of elements of P:

$$\operatorname{conv} P = \left\{ \sum_{i=1}^{m} r_i v_i \mid v_i \in P \text{ and } 0 \leqslant r_i \in \mathbb{R} \text{ with } \sum_{i=1}^{m} r_i = 1 \right\}.$$

Such a set is convex if $P = \operatorname{conv} P$.

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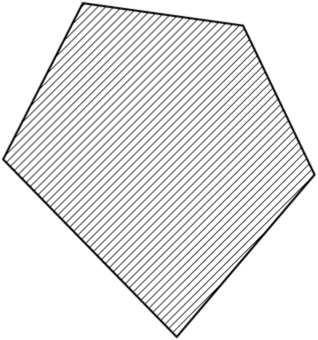
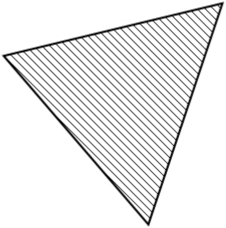
Such a set is convex if $P = \text{conv } P$.

The set P is called:

- a polytope, if there is a finite $F \subseteq \mathbb{R}^n$ with $P = \text{conv } F$;
- a rational polytope, if there is a finite $F \subseteq \mathbb{Q}^n$ with $P = \text{conv } F$.

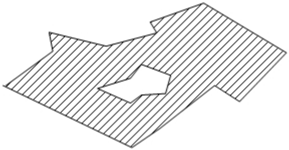


A polytope in $\mathbb{R}^2$.



A polytope in $\mathbb{R}^2$ (*a* simplex)*.*

A (compact) polyhedron in $\mathbb{R}^n$ is a union of finitely many polytopes in $\mathbb{R}^n$.

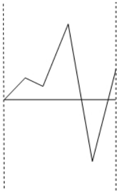


A polyhedron in $\mathbb{R}^2$*.*

Similarly, a rational polyhedron is a union of finitely many rational polytopes.

Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron. A continuous function $f: P \to \mathbb{R}$ is a $\mathbb{Z}$-map if the following hold.

1. There is a finite set $\{L_1, \ldots, L_m\}$ of affine linear functions $L_i: \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = L_{i_x}(x)$ for some $1 \leqslant i_x \leqslant m$.



A piecewise linear function $[0, 1] \to \mathbb{R}$.

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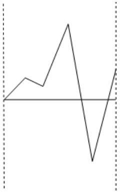
2. Each L_i can be written as a linear polynomial with integer coefficients.



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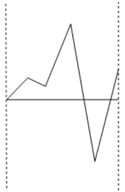

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A map $F\colon P \subseteq \mathbb{R}^n \to Q \subseteq \mathbb{R}^m$ between polyhedra always is of the form $F = (f_1, \ldots, f_m)$, $f_i\colon P \to \mathbb{R}$. Then F is a $\mathbb{Z}$-map if each one of its scalar components f_i is.

Rational polyhedra are precisely the subsets of $\mathbb{R}^n$ that are definable by a term in the language of MV-algebras; and $\mathbb{Z}$-maps are precisely the continuous transformations that are definable by tuples of terms in that language.

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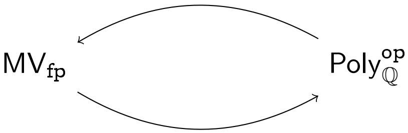
Stone-type duality for finitely presented MV-algebras

The category of finitely presented MV-algebras, and their homomorphisms, is equivalent to the opposite of the category of rational polyhedra, and the $\mathbb{Z}$-maps amongst them.

- V.M. & L. Spada, *Duality, projectivity, and unification in Łukasiewicz logic and MV-algebras*, Annals of Pure and Applied Logic, 2012.



From MV-algebras to rational polyhedra: Given $\mathscr{F}_n/\langle\tau(x_1,\ldots,x_n)\rangle$, the associated rational polyhedron $\mathbb{V}(\tau)$ is the set of n-tuples $(r_1,\ldots,r_n)\in[0,1]^n$ such that $\tau(r_1,\ldots,r_n)=0$ in $[0,1]$.

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Remark. The subspace $\mathbb{V}(\tau)\subseteq[0,1]^n$ homeomorphic to the maximal spectral space of $\mathscr{F}_n/\langle\tau\rangle$, topologised by the (analogue of) the Zariski topology. The MV-algebra $\nabla(P)$ is the exact analogue for rational polyhedra of the coordinate ring of an affine algebraic variety.

Syntax: Equations, or Formulæ — *Algebraic*

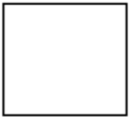
$$X \vee \neg X = 1$$

$$X \vee \neg X \vee Y \vee \neg Y = 1$$

Semantics: Solutions, or Models — *Geometrical*

$$X \vee \neg X = 1$$

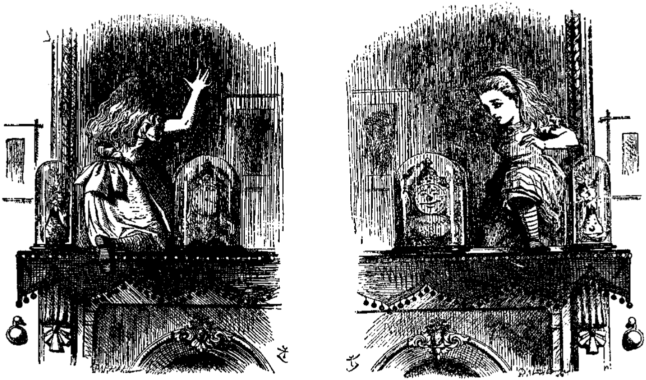
$$X \vee \neg X \vee Y \vee \neg Y = 1$$



Stone-type duality for finitely presented MV-algebras

$$\mathsf{MV}_{\mathrm{fp}} \rightleftarrows \mathsf{Poly}^{\mathrm{op}}_{\mathbb{Q}}$$

Epilogue



$$\mathsf{BoolAlg} \rightleftarrows \mathsf{StoneSp}^{\mathrm{op}}$$

In the thirties, Stone discovered that the set of maximal ideals of a Boolean algebra carries a natural topology: open sets correspond to arbitrary ideals. In the Introduction to his book on Stone spaces, P. Johnstone writes:

> *Now this was a really bold idea. Although the practitioners of abstract general topology* [...] *had by the early thirties developed considerable expertise in the construction of spaces with particular properties, the motivation of the subject was still geometrical* [...] *and* (*as far as I know*) *nobody had previously had the idea of applying these techniques to the study of spaces constructed from purely algebraic data.*

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The ensuing spaces are nowadays called Stone spaces. The clopen sets — those sets which are both closed and open in the topology — correspond to principal ideals/filters, and hence to elements of the algebra. Thus, the original algebra can be recovered from its space of maximal ideals; Stone's construction is in fact a two-way road.

The syntax-semantics dictionary.

Algebra, or Syntax.	Topology, or Semantics.
Boolean algebra	Stone (or Boolean) space
Homomorphism	Continuous map
Finite Boolean algebra	Finite set
Finite algebra homomorphism	Function
Free n-gen. algebra	$\{0,1\}^n$
Maximal ideal	Point of Stone space
Ideal	Closed subset of Stone space
Principal ideal	Clopen subset of Stone space
$\vdots$	$\vdots$

$$\mathsf{BoolAlg} \rightleftarrows \mathsf{StoneSp}^{\mathrm{op}}$$

The syntax-semantics dictionary.

Algebra, or Syntax.	Geometry, or Semantics.
F.p. algebra	Rational polyhedron
Homomorphism	$\mathbb{Z}$-map
F.p. subalgebra	Continuous image by $\mathbb{Z}$-map
F.p. quotient algebra	Rational subpolyhedron
F.p. projective algebra	Retract of cube by $\mathbb{Z}$-maps
Free n-gen. algebra	$[0,1]^n$
Maximal congruence	Point of rational polyhedron
Intersection of maximal cong.	Closed subset of rational polyhedron
Finite product $A \times B$	Finite disjoint union
$\vdots$	$\vdots$

$$\mathsf{MV}_{\mathrm{fp}} \rightleftarrows \mathsf{Poly}^{\mathrm{op}}_{\mathbb{Q}}$$

Algebraic geometry	General algebra
Ground field k $k[x_1, \ldots, x_n]$ Affine space k^n	A $\mathscr{F}_n$ A^n
Ideal of $k[x_1, \ldots, x_n]$ Affine variety in k^n Coord. ring $k[x_i]/\mathbb{I}(\mathbb{V}(S))$	Congruence on $\mathscr{F}_n$ Galois-fixed subset of A^n Quotient $\mathscr{F}_n/\mathbb{I}(\mathbb{V}(S))$
Homomorphism of k-alg. Map of affine varieties	V-homomorphism Term-definable map
Nullstellensatz co-*Nullstellensatz*	V.M. & L. Spada, 2012 ?
Maximal ideal ⋮	Maximal congruence ⋮

- V.M. and L. Spada, *The Dual Adjunction between MV-algebras and Tychonoff spaces*, Studia Logica 100, *in memoriam* Leo Esakia, 2012.
- O. Caramello, V.M., and L. Spada, *General affine adjunctions, and Nullstellensätze*, preliminary arXiv version, 2014.

Thank you for your attention.