Descriptive set theory

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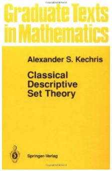
Introduction

1. 1870–1920: early developments: Cantor, Baire Borel, Lebesgue, Lusin, Suslin, Novikoff, ...
2. 1960–1990: the logicians' contribution: Addison, Solovay, Martin, Moschovakis, Kechris, Steel, Woodin, ...
3. 1990–now: interactions with other parts of mathematics: Kechris, Louveau, Becker, Hjorth, ...

For the most part, I will focus on parts 1 and 3.

Reference
A.S. Kechris
Classical Descriptive Set Theory
Springer 1995



The product topology

If $(X_i, \mathfrak{T}_i)$ are topological spaces, the product topology on $X = \prod_{i \in I} X_i$ is induced by the projections $X \twoheadrightarrow X_i$, $(x_j)_{j \in I} \mapsto x_i$. The basic open sets of $\prod_{i \in I} X_i$ are of the form

$$U_{i_1} \times \cdots \times U_{i_n} \times \prod_{i \in I \setminus \{i_1, \ldots, i_n\}} X_i$$

where $U_{i_k} \in \mathfrak{T}_{i_k}$, $(k = 1, \ldots, n)$. When I is a finite set or I is a countable infinite set, then a basis for $\prod_{i \in I} X_i$ has a particularly simple form: if $I = \{i_1, \ldots, i_n\}$, then a basis is obtained by taking sets of the form $U_{i_1} \times \cdots \times U_{i_n}$ with $U_{i_k} \in \mathfrak{T}_{i_k}$, $(k \in \{1, \ldots, n\})$; if $I = \mathbb{N}$, then a basis is obtained by taking sets of the form

$$U_0 \times \cdots \times U_n \times \prod_{i > n} X_i,$$

with $U_k \in \mathfrak{T}_k$, $(k \leq n)$. If $X_k = X$ for all k, write $X^{\mathbb{N}}$ instead of $\prod_k X$.

The product topology

A space is **separable** means that there is a countable dense set.

Proposition

The countable product of separable spaces is separable.

Proof.

For simplicity, take $X_n = X$, for all n, and let $D \subseteq X$ be countable and dense. Then $\left\{(x_n)_n \in D^{\mathbb{N}} \mid (x_n)_n \text{ is eventually constant}\right\}$ is dense in $X^{\mathbb{N}}$ □

The product topology

A space is **metrizable** if there is a metric which is compatible with the topology; it is **completely metrizable** if the metric can be taken to be complete.

Proposition

If the X_ns are (completely) metrizable, then $\prod_n X_n$ is (completely) metrizable.

Proof.

Suppose d_n is a (complete) metric on X_n. If needed, replace d_n with $d_n/(1+d_n)$ so that $d_n \leq 1$. Then

$$d\big((x_n)_n, (y_n)_n\big) = \sum_{n=0}^{\infty} \frac{d_n(x_n, y_n)}{2^n}$$

is a compatible (complete) metric. □

Polish spaces

Definition

A topological space is **Polish** if it separable and completely metrizable.

There may be many different complete metrics witnessing that X is Polish; if d is such a metric, then (X, d) is a **Polish metric space**.

The countable product of Polish spaces is Polish

A subset of a topological space is $\mathbf{G}_\delta$ if it is of the form $\bigcap_n U_n$ with U_n open sets.

A subspace of a Polish space is Polish iff it is $\mathbf{G}_\delta$.

Examples of Polish spaces

- $\mathbb{R}^n$, $\mathbb{R}^{\mathbb{N}}$, any separable Banach space.
- $[0;1]$, $(0;1)$, $\mathbb{R} \setminus \mathbb{Q}$.
- $\mathbf{K}(X) = \{K \subseteq X \mid K \text{ is compact}\}$ with (X,d) Polish. The metric is

  $$d_{\mathrm{H}}(K_1, K_2) = \max(\max\{d(x, K_2) \mid x \in K_1\}, \max\{d(x, K_1) \mid x \in K_2\})$$

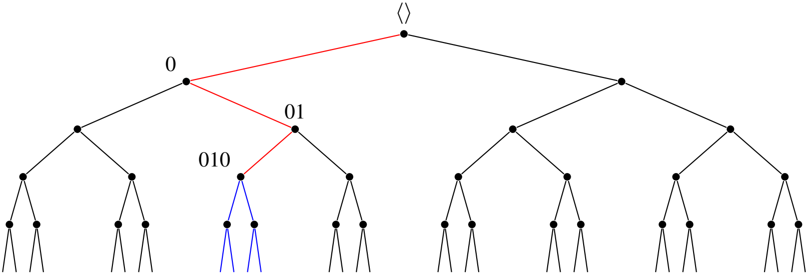
  where $d(x, K) = \min\{d(x,y) \mid y \in K\}$. If $D \subseteq X$ is countable dense, then $\{F \subseteq D \mid F \text{ is finite}\}$ is dense in $\mathbf{K}(X)$.
- K a compact metric space,
  $\mathcal{H}(X) = \{f \mid f\colon X \to X \text{ is a homeomorphism}\}$.
- X Polish and $\mathbb{P}(X) = \{\mu \mid \mu \text{ a Borel probability measure on } X\}$.
- Any countable set I with the discrete topology. Hence also $I^{\mathbb{N}}$ is Polish.

The Cantor and Baire spaces

If $2 = \{0, 1\}$ and $\mathbb{N}$ are given the discrete topology, then $2^{\mathbb{N}}$ the **Cantor space** and $\mathbb{N}^{\mathbb{N}}$ the **Baire space** are Polish. The basic open sets are the $\boldsymbol{N}_s$ with s a finite sequence

$$\boldsymbol{N}_s = \{(x_n)_n \mid \forall i < \mathrm{lh}(s)\ (s_i = x_i)\}$$

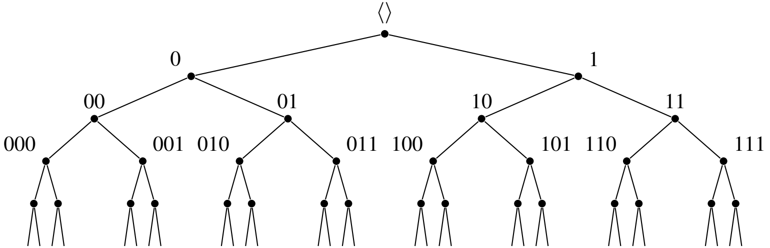
In $2^{\mathbb{N}}$, if $s = 010$, then $\boldsymbol{N}_s = \{x \in 2^{\mathbb{N}} \mid s \subseteq x\}$ is



Trees

A space that has a basis formed by clopen sets is called **totally disconnected** or **zero-dimensional**.

A **tree** on a set X is $T \subseteq X^{<\mathbb{N}}$, a collection of finite strings of elements from X, such that $t \in T \wedge s \subseteq t \Rightarrow s \in T$. For example



is the binary tree $2^{<\mathbb{N}}$.

$[T] = \{(x_n)_n \in X^{\mathbb{N}} \mid (x_0, \ldots, x_n) \in T$ for all $n \in \omega\}$ is the **body** of T. It is a topological space with the basic open neighborhoods given by N_s.

$[T]$ is compact iff T is finitely branching.

Zero-dimensional spaces

Closed subsets of $[T]$ are exactly of the form $[S]$ with S a subtree of T.
The metric on $[T]$ is given by $d(x,y) = 2^{-n}$, where n is least such that $x(n) \neq y(n)$.
This is an **ultrametric**, i.e. it satisfies

$$d(x,z) \leq \max(d(x,y), d(y,z))$$

In an ultrametric space, the open balls are **clopen**, that is they are both open and closed and every point is the center.
An ultrametric space is **zero-dimensional**, that is it has a basis consisting of clopen sets.
In particular, all this applies to the Cantor space $2^{\mathbb{N}} = \left[2^{<\mathbb{N}}\right]$ and to the Baire space $\mathbb{N}^{\mathbb{N}} = \left[\mathbb{N}^{<\mathbb{N}}\right]$.

The Cantor space

$2^{\mathbb{N}}$ is homeomorphic to $E_{1/3}$, the usual Cantor subsets of $[0;1]$, and to $\mathbb{Z}_p$, the ring of the p-adic integers.

Theorem (Brower)

$2^{\mathbb{N}}$ is the unique zero-dimensional compact separable space without isolated points.

Theorem (Cantor)

If C is a closed subset of a Polish space, then either C is countable, or else there is a continuous injective map $j\colon 2^{\mathbb{N}} \to C$.

Theorem

Every nonempty compact metric space is the continuous surjective image of $2^{\mathbb{N}}$.

The Cantor space

Proof.

If $K \neq \emptyset$ is compact metric, then it is homeomorphic to a closed subset C of $[0;1]^{\mathbb{N}}$. The function

$$f\colon 2^{\mathbb{N}} \to [0;1], \quad f(s) = \sum_{n=0}^{\infty} \frac{s(n)}{2^{n+1}}$$

is a continuous surjection, hence $(2^{\mathbb{N}})^{\mathbb{N}} \to [0;1]^{\mathbb{N}}$, $(s_k)_k \mapsto (f(s_k))_k$ is a continuous surjection. But $(2^{\mathbb{N}})^{\mathbb{N}} \approx 2^{\mathbb{N}}$, so we have a continuous surjection $2^{\mathbb{N}} \to [0;1]^{\mathbb{N}}$. Hence there is a closed set $[T] \subseteq 2^{\mathbb{N}}$ that subjects onto C. Construct a continuous surjection $\pi\colon 2^{\mathbb{N}} \to [T]$ so that π is the identity on $[T]$: this yields the desired surjection. □

The Baire space

Theorem (Alexandroff-Urysohn)

$\mathbb{N}^{\mathbb{N}}$ *is the unique zero-dimensional separable Polish space without isolated points, such that the compact sets have empty interior.*

Thus $\mathbb{N}^{\mathbb{N}}$ is far from being compact, in fact it is not even $\mathbf{K}_\sigma$.

Theorem (Hurewicz)

A Polish space is either $\mathbf{K}_\sigma$*, i.e., countable union of compact spaces, or else it contains a closed subset homeomorphic to* $\mathbb{N}^{\mathbb{N}}$*.*

Theorem

Every Polish space is the continuous injective image of some closed subset of $\mathbb{N}^{\mathbb{N}}$*.*

$\mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$.

Borel sets

- A **σ-algebra** on a set X is a family $\mathcal{S} \subseteq \mathscr{P}(X)$ closed under complements, countable unions and intersections.
- A **measurable space** is a pair $(X, \mathcal{S})$ with $\mathcal{S}$ a σ-algebra on X.
- The smallest σ-algebra containing all open subsets of a topological space X is the family of **Borel sets** $\mathrm{Bor}(X)$.
- A measurable space $(X, \mathcal{S})$ is a **standard Borel space** if there is a Polish topology on X such that $\mathcal{S} = \mathrm{Bor}(X)$.

$f\colon (X, \mathcal{S}) \to (Y, \mathcal{T})$ is measurable if $f^{-1}(A) \in \mathcal{S}$ for all $A \in \mathcal{T}$. When dealing with standard Borel spaces, we say that f is Borel. A Borel isomorphism between standard Borel spaces $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ is a **Borel isomorphism** if it is a bijection and both f and f^{-1} are measurable.

Borel isomorphisms

Theorem

Any two uncountable standard Borel spaces are Borel isomorphic. In particular, any two Polish spaces are Borel isomorphic.

Example

$\mathbf{F}(X) = \{F \subseteq X \mid F \text{ is closed}\}$ is standard Borel, with the **Effros-Borel σ-algebra** $\mathrm{EB}(X)$ generated by the sets $\{C \in \mathbf{F}(X) \mid C \cap U \neq \emptyset\}$, with U open.

Clearly, if X is compact, then $\mathbf{F}(X) = \mathbf{K}(X)$.
Let X be Polish:

- $\mathbf{K}(X)$ is Borel in $\mathbf{F}(X)$,
- the operation $\mathbf{F}(X) \times \mathbf{F}(X) \to \mathbf{F}(X)$, $(F_1, F_2) \mapsto F_1 \cup F_2$ is Borel.
- the operation $\mathbf{F}(X) \times \mathbf{F}(X) \to \mathbf{F}(X)$, $(F_1, F_2) \mapsto F_1 \cap F_2$ is *not* Borel.

Change of topology

Theorem

Let $A \in \mathrm{Bor}(X)$ with $(X, \mathfrak{T})$ a Polish space. Then there is a Polish topology $\mathfrak{T}' \supseteq \mathfrak{T}$ such that A is clopen in $\mathfrak{T}'$. Moreover $\mathfrak{T}'$ can be taken to be zero-dimensional.

Corollary

If $(X, \mathcal{S})$ is standard Borel and $A \in \mathcal{S}$ then either A is countable, or else there is a continuous injective map $j\colon 2^{\mathbb{N}} \to C$.

Proof.

Suppose A is uncountable. Change the topology so that A is clopen (in fact closed is enough): then A is Polish, so by Cantor's theorem there is a continuous $j\colon 2^{\mathbb{N}} \to A$. Since the new topology is finer, then j is still continuous with the older topology. □

The Borel hierarchy

If X is Polish let

- $\mathbf{\Sigma}^0_1$ be the set of open subsets, and $\mathbf{\Pi}^0_1$ be the set of closed subsets
- $\mathbf{\Sigma}^0_2 = \mathbf{F}_\sigma$ be the collection of all sets of the form $\bigcup_n C_n$ with C_n closed, and $\mathbf{\Pi}^0_2 = \mathbf{G}_\delta$ be the collection of all sets of the form $\bigcap_n U_n$ with U_n open,
- $\mathbf{\Sigma}^0_\alpha$ be the collection of all sets of the form $\bigcup_n A_n$ with $A_n \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta$, and $\mathbf{\Pi}^0_\alpha$ be the collection of all sets of the form $\bigcap_n A_n$ with $A_n \in \bigcup_{\beta<\alpha} \mathbf{\Sigma}^0_\beta$,

$\mathbf{\Delta}^0_\alpha \overset{\text{def}}{=} \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^0_\alpha = \left\{X \setminus A \mid A \in \mathbf{\Pi}^0_\alpha\right\}$, and

$$\mathrm{BOR} = \mathbf{\Sigma}^0_{\omega_1} = \mathbf{\Pi}^0_{\omega_1} = \mathbf{\Delta}^0_{\omega_1} = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Delta}^0_\alpha.$$

The Borel hierarchy

$$\begin{array}{ccccccccccccc}
 & & \boldsymbol{\Sigma}^0_1 & & & & \boldsymbol{\Sigma}^0_2 & & & & & \boldsymbol{\Sigma}^0_\alpha & \\
 & \subseteq & & \supseteq & & \subseteq & & & & & \subseteq & & \\
\boldsymbol{\Delta}^0_1 & & & & \boldsymbol{\Delta}^0_2 & & & & \cdots & \boldsymbol{\Delta}^0_\alpha & & & \cdots\cdots \\
 & \subseteq & & \supseteq & & \subseteq & & & & & \subseteq & & \\
 & & \boldsymbol{\Pi}^0_1 & & & & \boldsymbol{\Pi}^0_2 & & & & & \boldsymbol{\Pi}^0_\alpha &
\end{array}$$

A set is **true** $\boldsymbol{\Sigma}^0_\alpha$ if it is in $\boldsymbol{\Sigma}^0_\alpha \setminus \boldsymbol{\Pi}^0_\alpha = \boldsymbol{\Sigma}^0_\alpha \setminus \boldsymbol{\Delta}^0_\alpha$.

Examples

- $\mathbb{Q}$ is true $\boldsymbol{\Sigma}^0_2$, hence $\mathbb{R} \setminus \mathbb{Q}$ is true $\boldsymbol{\Pi}^0_2$ (Baire category theorem).
- $\left\{x \in 2^{\mathbb{N}\times\mathbb{N}} \mid \forall n \exists m \forall k > m\, (x(n,k) = 0)\right\}$ is true $\boldsymbol{\Pi}^0_3$.
- $c_0 = \{(x_n)_n \in \mathbb{R}^{\mathbb{N}} \mid \lim_n x_n = 0\}$ is true $\boldsymbol{\Pi}^0_3$.
- $\mathcal{C}^\infty(\mathbb{S}^1) \subset \mathcal{C}(\mathbb{S}^1, \mathbb{R})$ is true $\boldsymbol{\Pi}^0_3$.

Property of Baire

Definition

X a topological space.
$M \subseteq X$ is **meager** if $M \subseteq \bigcup_n C_n$ with C_n closed and with empty interior.
The complement of a meager set is **comeager**.
$A \subseteq X$ has the **property of Baire** if $A \triangle U$ is meager, for some open set U.

$\textsc{Mgr}$ is the collection of meager sets, and $\textsc{Bp}$ is the collection of all sets with the property of Baire.

Theorem (Baire)

If X is completely metrizable or compact (in particular: if X is Polish), then no non-empty open set is meager.

$\textsc{Bp}$ is a σ-algebra extending $\textsc{Bor}$.

Projective sets

X Polish and $A \subseteq X$:

- A is **analytic** or $\boldsymbol{\Sigma}^1_1$ if it is the continuous (or even Borel) image of a Borel set. It is **coanalytic** or $\boldsymbol{\Pi}^1_1$ if $X \setminus A \in \boldsymbol{\Sigma}^1_1$.
- A is $\boldsymbol{\Sigma}^1_{n+1}$ if it is the continuous (or even Borel) image of a $\boldsymbol{\Pi}^1_n$ set. It is $\boldsymbol{\Pi}^1_{n+1}$ if $X \setminus A \in \boldsymbol{\Sigma}^1_{n+1}$.

$\boldsymbol{\Delta}^1_n = \boldsymbol{\Sigma}^1_n \cap \boldsymbol{\Pi}^1_n$ and the projective sets are the elements of $\bigcup_n \boldsymbol{\Sigma}^1_n = \bigcup_n \boldsymbol{\Pi}^1_n = \bigcup_n \boldsymbol{\Delta}^1_n$.

Theorem

(Suslin) If A and B are disjoint $\boldsymbol{\Sigma}^1_1$, then there is a Borel set C such that $A \subseteq C$ and $B \cap C = \emptyset$.
In particular: (Lusin) $\boldsymbol{\Delta}^1_1 = \mathrm{Bor}$.

Proposition

The continuous (or even Borel) injective image of a Borel set is Borel.

The projective hierarchy

$$
\begin{array}{ccccccccccc}
 & & \boldsymbol{\Sigma}^1_1 & & & & \boldsymbol{\Sigma}^1_2 & & & & \boldsymbol{\Sigma}^1_n & \\
 & \subseteq & & \supseteq & & \subseteq & & & & \subseteq & & \\
\mathrm{BOR} = \boldsymbol{\Delta}^1_1 & & & & \boldsymbol{\Delta}^1_2 & & & \cdots & \boldsymbol{\Delta}^1_n & & & \cdots\cdots \\
 & \subseteq & & \supseteq & & \subseteq & & & & \subseteq & & \\
 & & \boldsymbol{\Pi}^1_1 & & & & \boldsymbol{\Pi}^1_2 & & & & \boldsymbol{\Pi}^1_n &
\end{array}
$$

A set is true $\boldsymbol{\Sigma}^1_n$ if it is in $\boldsymbol{\Sigma}^1_n \setminus \boldsymbol{\Pi}^1_n$, and similarly for true $\boldsymbol{\Pi}^1_n$.

Examples

- If $\mathfrak{X}$ is a separable Banach space and $T \in L(\mathfrak{X})$ then $\mathrm{spec}(T) = \{\lambda \in \mathbb{C} \mid \exists x \neq 0 (T(x) = \lambda x)\}$ is $\boldsymbol{\Sigma}^1_1$ and bounded. Every bounded $\boldsymbol{\Sigma}^1_1$ subset of $\mathbb{C}$ is of this form.
- The set $\mathcal{D} \subseteq \mathcal{C}([0;1])$ of all differentiable functions is true $\boldsymbol{\Pi}^1_1$.
- $\{T \in L(c_0) \mid \mathrm{spec}(T) = \mathbb{S}^1\}$ is true $\boldsymbol{\Pi}^1_2$.

Foundational issues

Classical results

1. Every $\mathbf{\Sigma}^1_1$, and hence every $\mathbf{\Pi}^1_1$ has the property of Baire, and it is Lebesgue measurable.
2. A $\mathbf{\Sigma}^1_1$ set is either countable (hence $\mathbf{F}_\sigma$) or else it contains a copy of Cantor, i.e. there is $j\colon 2^{\mathbb{N}} \to A$ injective and continuous.

Question

Do these properties hold for all projective sets?

Consistently false

(Gödel, 1938) In the constructible universe L there is a $\mathbf{\Delta}^1_2$ set that is not Lebesgue measurable and that does not have the Baire property, and there is an uncountable $\mathbf{\Pi}^1_1$ set that does not contain a copy of the Cantor set.

Foundational issues

Consistently true (modulo modest large cardinals)

(Solovay 1965) There is a model of ZFC where all uncountable projective sets are Lebesgue measurable, have the property of Baire, and contain a copy of $2^{\mathbb{N}}$.

True (modulo large cardinals)

If there is a measurable cardinal (Solovay, 1968) then all sets in $\boldsymbol{\Sigma}^1_2 \cup \boldsymbol{\Pi}^1_2$ are Lebesgue measurable, $\boldsymbol{\Pi}^1_1 \subseteq \mathrm{B}\textsc{p}$, and every uncountable $\boldsymbol{\Pi}^1_1$ contains a copy of $2^{\mathbb{N}}$.
If there are infinitely many Woodin cardinals (Martin-Steel 1988) all uncountable projective sets are Lebesgue measurable, have the property of Baire, and contain a copy of $2^{\mathbb{N}}$.

Methods of logic, like recursion theory, forcing, large cardinals, definability theory, and most importantly infinite games play a key role here.

Polish groups

Definition

A group G with a Polish topology such that $(x, y) \mapsto xy^{-1}$ is continuous is a **Polish group**.

$\mathbb{R}^n$, $\mathbb{R}^{\mathbb{N}}$, separable Banach spaces, are examples of Polish groups.

Any metrizable group has a **left-invariant metric** d, that is $d(zx, zy) = d(x, y)$, and similarly a **right-invariant metric**. Abelian groups have metrics which are both-sides invariant.

A Polish group has a complete metric, and a left-invariant one, but needs not to have a complete *and* left-invariant metric. For example $S_\infty = \{f \in \mathbb{N}^{\mathbb{N}} \mid f \text{ is a bijection}\}$ is a group, it is $\mathbf{G}_\delta$ in $\mathbb{N}^{\mathbb{N}}$ hence it is a Polish space, the usual metric d on $\mathbb{N}^{\mathbb{N}}$ is left-invariant, but not complete. A complete metric on S_∞ is given by $D(f, g) = d(f, g) + d(f^{-1}, g^{-1})$.

Polish groups

Subgroups are closed

If G is Polish, $H \leq G$, and H is Polish (i.e. $\mathbf{G}_\delta$), then H is closed.

Uniqueness of the topology

Suppose G is a group, and also a standard Borel space, and that $(x, y) \mapsto xy^{-1}$ is Borel. Then there is *at most* one Polish topology that makes G a Polish group.

Automatic continuity

Let $f \colon G \to H$ be a homomorphism of Polish group, and suppose it is **Baire measurable**, i.e. the preimage of an open set is in BP. Then f is continuous.

The measure algebra

For X a standard Borel space and μ a Borel measure on X, the **measure algebra** $\mathrm{MALG}(X) = \mathrm{BOR}(X)/\mathrm{NULL}$ is the Borel sets, modulo μ-measure 0 sets. It is a Polish group with distance $d([A],[B]) = \mu(A \triangle B)$, and operation $[A] + [B] = [A \triangle B]$.

Fact

$\mathrm{MALG}(X)$ does not depend on X or on μ, as long as $\mu(x) = 0$ for all $x \in X$.

In other words: the measure algebra on $\mathbb{R}$ with the Lebesgue measure or on $2^{\mathbb{N}}$ with the coin-tossing measure are isomorphic.

How to parametrize objects

- Every compact metric space is homeomorphic to a closed subspace of $[0;1]^{\mathbb{N}}$. Hence $\mathbf{K}([0;1]^{\mathbb{N}})$ is the Polish space of all compact metric spaces.
- Every Polish space is homeomorphic to a closed subset of $\mathbb{R}^{\mathbb{N}}$. Hence $\mathbf{F}(\mathbb{R}^{\mathbb{N}})$ is the standard Borel space of all Polish spaces.
- Every separable Banach space is homeomorphic to a closed subspace of $\mathscr{C}[0;1]$, the Banach space of all real valued continuous functions on $[0;1]$. The set $\mathscr{B} \subseteq \mathbf{F}(\mathscr{C}[0;1])$ of all separable Banach space is Borel.
- Every Polish group is (isomorphic to) a closed subgroup of $\mathcal{H}([0;1]^{\mathbb{N}})$, the homeomorphisms of the Hilbert cube. The set $\mathscr{G} \subseteq \mathbf{F}(\mathcal{H}([0;1]^{\mathbb{N}}))$ of all Polish groups is Borel.

Equivalence relations

X a standard Borel space and E an equivalence relation on X which, as a subset of $X \times X$ is Borel or projective of low level (analytic or co-analytic).

Examples

- $=_X$ equality on X is closed, hence Borel.
- G Polish group, $a\colon G \times X \to X$ a Borel action on a standard Borel X. Then E_G is $\boldsymbol{\Sigma}^1_1$: $x \; E_G \; y \Leftrightarrow \exists g \in G \; (a(g,x) = y)$.
- Consider the relation of **Turing equivalence** on $\mathscr{P}(\mathbb{N}) = 2^{\mathbb{N}}$: $x =_T y$ just in case x and y can be computed from each other. Then $=_T$ is a Borel equivalence relation on $2^{\mathbb{N}}$, each equivalence class is countable, and the quotient $\mathcal{D}$ is the upper semi-lattice of Turing degrees.

Classification

If E is on X and F on Y then set

$$E \leq_B F \Leftrightarrow \exists f : X \to Y \text{ Borel } \forall x_1, x_2 \in X \ (x_1 \ E \ x_2 \Leftrightarrow f(x_1) \ F \ f(x_2)) \, .$$

This implies that $|X/E| \leq |Y/F|$.

Theorem (Silver)

E coanalytic equivalence relation on a Polish space X, then

- *either $E \leq_B =_{\mathbb{N}}$, i.e. X/E is countable,*
- *or else there is $C \subseteq X$ homeomorphic to Cantor, of pairwise E-inequivalent elements, hence $=_{2^{\mathbb{N}}} \leq_B E$.*

Countable Borel equivalence relations

Theorem (Feldman-Moore)

If E is a ***countable Borel equivalence relation*** *on a standard Borel space X, i.e. such that all equivalence classes are at most countable, then $E = E_G$ with G a countable group acting in a Borel way on X.*

Theorem (Jackson-Kechris-Louveau)

There is a countable Borel equivalence relation E_∞ such that $E \leq_B E_\infty$ for all countable Borel equivalence relations.

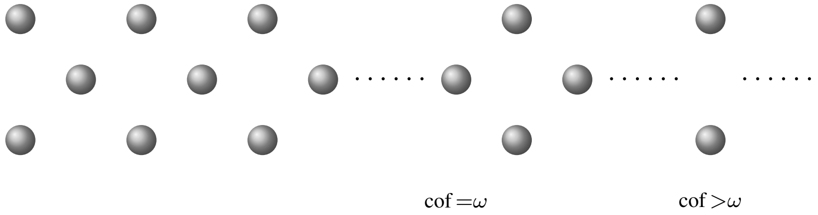
The structure of the ordering $\leq_B$ even on countable Borel equivalence relations is extremely complex.
The study of Borel and analytic equivalence relations on standard Borel spaces is one of the main topics in Descriptive Set Theory.

The Wadge hierarchy

The Borel hierarchy on $\mathbb{N}^{\mathbb{N}}$ (and more generally on every zero-dimensional Polish space, like $2^{\mathbb{N}}$) admits an ultimate refinement. For $A, B \subseteq \mathbb{N}^{\mathbb{N}}$, say that A is Wadge-riducible to B, in symbols $A \leq_{\mathrm{W}} B$ iff

$$A = f^{-1}(B), \text{ for some continuous } f \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$$

$A \equiv_{\mathrm{W}} B \Leftrightarrow A \leq_{\mathrm{W}} B \leq_{\mathrm{W}} A$ and the equivalence classes are the Wadge degrees.



Infinite games play a key role here!

Completeness

Let $\mathbf{\Gamma}$ be one of the $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$.
A set $Y \subseteq \mathbb{N}^\mathbb{N}$ is **$\mathbf{\Gamma}$-complete** if it is in $\mathbf{\Gamma}$ and $Z \leq_{\mathrm{W}} Y$ for all $Z \subseteq \mathbb{N}^\mathbb{N}$ in $\mathbf{\Gamma}$.

Any $Y \subseteq \mathbb{N}^\mathbb{N}$ is true $\mathbf{\Gamma}$ if and only if it is $\mathbf{\Gamma}$-complete.

This can be proved outright in ZFC if $\mathbf{\Gamma} \subseteq \mathrm{Bor}$, and assuming large cardinals it holds also for projective sets.
Let X be an arbitrary Polish space. To show that $A \subseteq X$ is true $\mathbf{\Gamma}$ it is enough to show that

- A is in $\mathbf{\Gamma}$, and
- there is a continuous $f \colon X \to \mathbb{N}^\mathbb{N}$ and $\mathbf{\Gamma}$-complete $B \subseteq \mathbb{N}^\mathbb{N}$ such that $A = f^{-1}(B)$.

By analogy, we will say that A is $\mathbf{\Gamma}$-complete.

The Lebesgue density theorem

Shameless self-promotion

This is joint work with Riccardo Camerlo (Politecnico di Torino)

λ Lebesgue measure on $\mathbb{R}$, $A \subseteq \mathbb{R}$ measurable, let

$$\Phi(A) = \{x \in \mathbb{R} \mid \mathscr{D}_A(x) = 1\}$$

where $\mathscr{D}_A(x) = \lim_{\varepsilon \to 0^+} \frac{\lambda(A \cap (x-\varepsilon; x+\varepsilon))}{2\varepsilon}$ is the **density** of A at x.
The Lebesgue density theorem says that $A \equiv \Phi(A)$, meaning $\lambda(A \triangle \Phi(A)) = 0$, hence Φ selects a representative in each equivalence class of MALG.
The Lebesgue density theorem holds also for $2^{\mathbb{N}}$ with the **coin-tossing measure** $\mu(\boldsymbol{N}_s) = 2^{-\operatorname{lh} s}$, and $\mathscr{D}_A(x) = \lim_{n\to\infty} \mu(A \cap \boldsymbol{N}_{x \restriction n}) \cdot 2^n$.
It is not hard to see that $\Phi(A) \in \boldsymbol{\Pi}^0_3$.

Question

What is the exact complexity of $\Phi(A)$?

The complexity of $\Phi(A)$ when the ambient space is $2^{\mathbb{N}}$

Theorem

For each Wadge degree up to $\mathbf{\Pi}^0_3$ there is an open (or a closed) set $B \subseteq 2^{\mathbb{N}}$ such that $\Phi(B)$ is in that degree, that is: for all $A \subseteq 2^{\mathbb{N}}$ in $\mathbf{\Pi}^0_3$ there is B as above such that $A \equiv_{\mathrm{W}} \Phi(B)$.

In particular, there is an open (or a closed) set $B \subseteq 2^{\mathbb{N}}$ such that $\Phi(B)$ is $\mathbf{\Pi}^0_3$-complete.

Theorem

The set of $[A] \in \mathrm{MALG}$ such that $\Phi(A)$ is true $\mathbf{\Pi}^0_3$ is comeager.
In fact, if $C \subseteq 2^{\mathbb{N}}$ has positive measure and empty interior in $2^{\mathbb{N}}$, then $\Phi(C)$ is $\mathbf{\Pi}^0_3$-complete.

The range of the density function

Work either in $2^{\mathbb{N}}$: the set

$$\operatorname{ran} \mathscr{D}_A = \left\{ r \in [0;1] \mid \exists x \in 2^{\mathbb{N}}\, \mathscr{D}_A(x) = r \right\}$$

is $\boldsymbol{\Sigma}^1_1$. (Note that the limit $\mathscr{D}_A(x)$ might not exist, for some x.)

Theorem

- *For every $\boldsymbol{\Sigma}^1_1$ set $S \subseteq (0;1)$ there is $C \subseteq 2^{\mathbb{N}}$ compact of positive measure such that $\operatorname{ran}(\mathscr{D}_C) = S \cup \{0,1\}$.*
- $\{C \in \mathbf{K}(2^{\mathbb{N}}) \mid \operatorname{ran}(\mathscr{D}_C) = \{0,1\}\}$ *is $\boldsymbol{\Pi}^1_1$-complete.*
- $\{C \in \mathbf{K}(2^{\mathbb{N}}) \mid \operatorname{ran}(\mathscr{D}_C) = [0;1]\}$ *is $\boldsymbol{\Pi}^1_2$-complete.*

$2^{\mathbb{N}}$ is disconnected and there are sets A such that $\mathscr{D}_A(x)$ is defined for all x and takes values only 0 and 1.
This fails badly for $\mathbb{R}^n$! Yet the theorem above still holds.


That's all Folks!