Ordered Algebras and Logic

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Logic and Mathematics

We know that mathematicians care no more for logic than logicians for mathematics. The two eyes of science are mathematics and logic; the mathematical sect puts out the logical eye, the logical sect puts out the mathematical eye; each believing that it sees better with one eye than with two.

AUGUSTUS DE MORGAN

Logic and Algebra

From the textbook definition of a group

> A *group* is an ordered pair $(G, \circ)$ such that G is a set, $\circ$ is an associative binary operation on G, and $\exists\, e \in G$ such that
>
> (i) if $a \in G$, then $a \circ e = a$,
>
> (ii) if $a \in G$, then $\exists\, a^{-1} \in G$ such that $a \circ a^{-1} = e$.

we can obtain a set of first-order sentences

$$\Gamma = \{(\forall x)(\forall y)(\forall z)(x \circ (y \circ z) \approx (x \circ y) \circ z), \\ (\exists x)(\forall y)(y \circ x \approx y \;\&\; (\exists z)(y \circ z \approx x))\}$$

and ask about its **consequences**, e.g.

$$\Gamma \models (\forall x)(\forall y)(\exists z)(x \circ z \approx y) \quad ?$$

$$\text{or} \qquad \Gamma \vdash (\forall x)(\forall y)(\exists z)(x \circ z \approx y) \quad ?$$

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Today

This tutorial will consist of two parts:

(I) Consequence in Logic and Algebra

(II) Substructural Logics and Residuated Lattices.

Part I

Consequence in Logic and Algebra

Categorical syllogisms, as described by Aristotle in the *Prior Analytics* (c. 350 BC), consist of three parts: the major premise, the minor premise, and the conclusion.

For example:

Major premise:	No homework is fun.	(No M are P.)
Minor premise:	Some reading is homework.	(Some S are M.)
Conclusion:	Some reading is not fun.	(Some S are not P.)

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Boolean algebras originated in George Boole's *An Investigation of the Laws of Thought* (1865) and consist of a set B with binary operations $\wedge, \vee$, a unary operation $'$, and constants $0, 1$.

Key examples include:

- the **two-element Boolean algebra** (with $x' = 1 - x$)

$$(\{0, 1\}, \min, \max,', 0, 1)$$

- **power set algebras**, for a set A (with $B' = A \setminus B$)

$$(\wp(A), \cap, \cup,', \emptyset, A).$$

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Formal systems for logical consequence (the predicate calculus) based on the notion of **proof** were developed by Frege, Hilbert, Bernays, Russell, Gentzen, and others (1879-1935).

$$\Gamma \vdash \varphi \quad \text{“There is a proof of } \varphi \text{ from } \Gamma\text{.”}$$

A "truth-oriented" description of logical consequence was given by Alfred Tarski (1936) based on **models**: mathematical structures that provide interpretations for non-logical primitives of a formal language.

$\Gamma \models \varphi$ "If $\mathbf{A}$ is a model of Γ, then $\mathbf{A}$ is a model of φ."

The equivalence of the semantic (truth) and syntactic (proof) approaches was established by Kurt Gödel in his 1929 doctoral dissertation, i.e.

$$\Gamma \vdash \varphi \qquad \Leftrightarrow \qquad \Gamma \models \varphi.$$

Investigating Consequence

- A more abstract framework for investigating consequence is provided by Tarski's notion of a **consequence relation**.
- We consider here how consequence relations can be defined in terms of **proof systems** and **classes of algebras**.
- We give an account (following Lindenbaum-Tarski, Blok-Pigozzi, Jónsson, etc.) of the **equivalence** of consequence relations.
- As an example, we consider equivalent consequence relations for the class of **lattices** and a simple application.

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Consequence Relations

A **consequence relation** over a non-empty set A is a relation $\vdash \subseteq \wp(A) \times A$ such that for all $a, b \in A$ and $X, Y \subseteq A$:

- $X \vdash a$ if $a \in X$ (reflexivity)
- $X \vdash a$ implies $X \cup Y \vdash a$ (monotonicity)
- $X \vdash a$ and $X \cup \{a\} \vdash b$ implies $X \vdash b$ (transitivity).

$\vdash$ is called **finitary** if also

- $X \vdash a$ implies $Y \vdash a$ for some finite $Y \subseteq X$.

We write $X \vdash Y$ when $X \vdash a$ for all $a \in Y$.

Note that consequence relations over A are in 1-1 correspondence with **consequence operators** (closure operators) on the poset $(\wp(A), \subseteq)$.

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Some Terminology

To talk about logics and classes of algebras, we need

- a **language** $\mathcal{L}$ consisting of function symbols (or connectives) such as $\circ$, $^{-1}$, e, $\wedge$, $\vee$, $\neg$, 0, 1 with specified finite arities
- **$\mathcal{L}$-algebras** consisting of a set A together with functions $f^{\mathbf{A}}$ for each function symbol f of $\mathcal{L}$
- the set $\mathrm{Fm}_{\mathcal{L}}$ of **$\mathcal{L}$-formulas** $\varphi, \psi \ldots$ built from a countably infinite set of variables $x, y \ldots$ and the **formula algebra** $\mathbf{Fm}_{\mathcal{L}}$
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A Proof System for Classical Logic

Let $\mathcal{L}$ be a language with connectives $\wedge, \vee, \to, \neg, 0, 1$ and define

$$\Gamma \vdash_{\text{HCL}} \varphi$$

when $\varphi \in \text{Fm}_{\mathcal{L}}$ is derivable from $\Gamma \subseteq \text{Fm}_{\mathcal{L}}$ using the (schematic) rules:

A1. $\varphi \to (\psi \to \varphi)$
A2. $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
A3. $\neg\neg\varphi \to \varphi$
A4. $(\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi)$
A5. $\varphi \to (\neg\varphi \to \psi)$
A6. $1 \to (\varphi \to \varphi)$
A7. $(\varphi \to \psi) \to ((\varphi \to \chi) \to (\varphi \to \psi \wedge \chi))$
A8. $(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
A9. $\varphi \wedge \psi \to \varphi$
A10. $\varphi \wedge \psi \to \psi$
A11. $\varphi \to \varphi \vee \psi$
A12. $\psi \to \varphi \vee \psi$
A13. $\neg 1 \to 0$
A14. $0 \to \neg 1$
A15. $(\varphi \to \varphi) \to 1$

$$\frac{\varphi \quad \varphi \to \psi}{\psi} \text{ (MP)}$$

Then $\vdash_{\text{HCL}}$ is a finitary consequence relation over $\text{Fm}_{\mathcal{L}}$.

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A3.	$\neg\neg\varphi \rightarrow \varphi$	A11.	$\varphi \rightarrow \varphi \vee \psi$
A4.	$(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi)$	A12.	$\psi \rightarrow \varphi \vee \psi$
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A8.	$(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \vee \psi \rightarrow \chi))$		

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Then $\vdash_{\mathrm{HCL}}$ is a finitary consequence relation over $\mathrm{Fm}_{\mathcal{L}}$.

Proof Systems

A **rule** for a set A is a set of ordered pairs $(\{a_1, \ldots, a_n\}, a)$ with $\{a_1, \ldots, a_n, a\} \subseteq A$, and a **proof system** $\mathcal{C}$ is a set of rules for A.

A $\mathcal{C}$**-derivation** of $a \in A$ from $X \subseteq A$ is a finite tree labelled with members of A such that a labels the root and each node labelled b

- is either in X
- or has child nodes labelled $b_1, \ldots, b_n$ where $(\{b_1, \ldots, b_n\}, b)$ is a member of a rule of $\mathcal{C}$.

We write $X \vdash_{\mathcal{C}} a$ if there is a $\mathcal{C}$-derivation of a from X.

Lemma

$\vdash_{\mathcal{C}}$ is a finitary consequence relation over A.

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A C**-derivation** of $a \in A$ from $X \subseteq A$ is a finite tree labelled with members of A such that a labels the root and each node labelled b

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Proof Systems

A **rule** for a set A is a set of ordered pairs $(\{a_1, \ldots, a_n\}, a)$ with $\{a_1, \ldots, a_n, a\} \subseteq A$, and a **proof system** $\mathcal{C}$ is a set of rules for A.

A $\mathcal{C}$**-derivation** of $a \in A$ from $X \subseteq A$ is a finite tree labelled with members of A such that a labels the root and each node labelled b

- is either in X
- or has child nodes labelled $b_1, \ldots, b_n$ where $(\{b_1, \ldots, b_n\}, b)$ is a member of a rule of $\mathcal{C}$.

We write $X \vdash_{\mathcal{C}} a$ if there is a $\mathcal{C}$-derivation of a from X.

Lemma

$\vdash_{\mathcal{C}}$ *is a finitary consequence relation over A.*

Substitution-Invariance

$\mathcal{L}$**-substitutions** for a language $\mathcal{L}$ can be defined as endomorphisms

$$\sigma \colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}$$

and extended to $\mathcal{L}$-equations by

$$\sigma(\varphi \approx \psi) = \sigma(\varphi) \approx \sigma(\psi).$$

A consequence relation $\vdash$ over $\mathrm{Fm}_{\mathcal{L}}$ or $\mathrm{Eq}_{\mathcal{L}}$ satisfying

$$X \vdash a \quad \Longrightarrow \quad \sigma(X) \vdash \sigma(a) \text{ for all } \mathcal{L}\text{-substitutions } \sigma$$

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Equational Consequence Relations

Given a class of $\mathcal{L}$-algebras $\mathcal{K}$, define for $\Sigma \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}$:

$\Sigma \vdash_{\mathcal{K}} \varphi \approx \psi \quad \Longleftrightarrow$ "whenever the equations in Σ hold in some $\mathbf{A} \in \mathcal{K}$, also $\varphi \approx \psi$ holds in $\mathbf{A}$"

For each $\mathbf{A} \in \mathcal{K}$ and $h \colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$

$$h(\varphi') = h(\psi') \text{ for all } \varphi' \approx \psi' \in \Sigma \quad \Longrightarrow \quad h(\varphi) = h(\psi).$$

Lemma

$\vdash_{\mathcal{K}}$ is a substitution-invariant consequence relation over $\mathrm{Eq}_{\mathcal{L}}$. Moreover, if $\mathcal{K}$ is a variety (equational class), then $\vdash_{\mathcal{K}}$ is finitary.

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Boolean Algebras

Recall that a **Boolean algebra** (in the same language as HCL) is an algebra $\mathbf{A} = (A, \wedge, \vee, \to, \neg, 0, 1)$ such that

- $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice
- $a \wedge \neg a = 0$ and $a \vee \neg a = 1$ for all $a \in A$
- $a \to b = \neg a \vee b$ for all $a, b \in A$.

Let $\mathcal{BA}$ be the (equational) class of all Boolean algebras. Then $\vdash_{\mathcal{BA}}$ is a finitary substitution-invariant consequence relation.

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Equivalence of $\vdash_{\mathrm{HCL}}$ and $\vdash_{\mathcal{BA}}$

We translate between formulas and equations using "transformers"

$$\tau(\varphi) = \{\varphi \approx 1\} \qquad \text{and} \qquad \rho(\varphi \approx \psi) = \{\varphi \to \psi, \psi \to \varphi\}$$

and obtain

$$\begin{array}{rcl} \Gamma \vdash_{\mathrm{HCL}} \varphi & \iff & \tau(\Gamma) \vdash_{\mathcal{BA}} \tau(\varphi) \\ \Sigma \vdash_{\mathcal{BA}} \varphi \approx \psi & \iff & \rho(\Sigma) \vdash_{\mathrm{HCL}} \rho(\varphi \approx \psi) \\ \varphi \vdash_{\mathrm{HCL}} \rho(\tau(\varphi)) & \& & \rho(\tau(\varphi)) \vdash_{\mathrm{HCL}} \varphi \\ \varphi \approx \psi \vdash_{\mathcal{BA}} \tau(\rho(\varphi \approx \psi)) & \& & \tau(\rho(\varphi \approx \psi)) \vdash_{\mathcal{BA}} \varphi \approx \psi. \end{array}$$

We say that $\mathcal{BA}$ is an "equivalent algebraic semantics" for $\vdash_{\mathrm{HCL}}$.

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Algebraizable Logics

A substitution-invariant consequence relation $\vdash_{\mathrm{L}}$ over $\mathrm{Fm}_{\mathcal{L}}$ is called **algebraizable** with respect to a class of $\mathcal{L}$-algebras $\mathcal{K}$ if there are maps

$$\tau\colon \mathrm{Fm}_{\mathcal{L}} \to \wp(\mathrm{Eq}_{\mathcal{L}}) \qquad \text{and} \qquad \rho\colon \mathrm{Eq}_{\mathcal{L}} \to \wp(\mathrm{Fm}_{\mathcal{L}})$$

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Examples

Logic	**Equivalent algebraic semantics**
Classical logic	Boolean algebras
Intuitionistic logic	Heyting algebras
Modal logics	Boolean algebras with operators
Łukasiewicz logic	MV-algebras
⋮	⋮
BCI logic	not algebraizable!

A More Abstract Perspective

Let $\vdash_1, \vdash_2$ be consequence relations over the sets A_1, A_2, respectively.

We say that $\vdash_1$ and $\vdash_2$ are **similar** if there exist maps

$$\tau\colon A_1 \to \wp(A_2) \qquad \text{and} \qquad \rho\colon A_2 \to \wp(A_1)$$

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a \vdash_1 \rho(\tau(a)) & \& & \rho(\tau(a)) \vdash_1 a \\
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Action-Invariance

A non-empty set A is called an **M-set** if there exists a monoid

$$\mathbf{M} = (M, \circ, 1)$$

and an operation

$$\star\colon M \times A \to A$$

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A consequence relation $\vdash$ over A is **action-invariant** if for all $\sigma \in M$:

$$X \vdash a \quad \Longrightarrow \quad \sigma \star X \vdash \sigma \star a.$$

Action-Invariance

A non-empty set A is called an **M-set** if there exists a monoid

$$\mathbf{M} = (M, \circ, 1)$$

and an operation

$$\star \colon M \times A \;\rightarrow\; A$$

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Equivalence

Let $\vdash_1$ and $\vdash_2$ be action-invariant consequence relations over **M**-sets A_1 and A_2, respectively.

We say that $\vdash_1$ and $\vdash_2$ are **equivalent** if

- $\vdash_1$ and $\vdash_2$ are similar with transformers τ and ρ
- τ can be extended to an action-invariant map

$$\tau^* \colon \wp(A_1) \to \wp(A_2)$$

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Lattices

Recall that a **lattice** is a poset $(L, \leq)$ containing for all $x, y \in L$

- $x \wedge y$: the greatest lower bound (meet) of x and y
- $x \vee y$: the least upper bound (join) of x and y

or, alternatively, an algebra $(L, \wedge, \vee)$ satisfying

$$\begin{array}{rclcrcl} x \vee (y \vee z) & \approx & (x \vee y) \vee z & \quad & x \wedge (y \wedge z) & \approx & (x \wedge y) \wedge z \\ x \wedge y & \approx & y \wedge x & & x \vee y & \approx & y \vee x \\ x \wedge (x \vee y) & \approx & x & & x \vee (x \wedge y) & \approx & x \end{array}$$

where $x \leq y$ stands for $x \wedge y \approx x$.

The class $\mathcal{LAT}$ of all lattices (as algebras) has a corresponding substitution-invariant finitary consequence relation $\vdash_{\mathcal{LAT}}$.

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A Proof System Lat for Lattices

Axioms

$$\frac{}{\varphi \leq \varphi}\ (\text{ID})$$

Cut rule

$$\frac{\varphi \leq \chi \quad \chi \leq \psi}{\varphi \leq \psi}\ (\text{CUT})$$

Left operational rules

$$\frac{\varphi_1 \leq \psi}{\varphi_1 \wedge \varphi_2 \leq \psi}\ (\wedge\Rightarrow)_1$$

$$\frac{\varphi_2 \leq \psi}{\varphi_1 \wedge \varphi_2 \leq \psi}\ (\wedge\Rightarrow)_2$$

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Right operational rules

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Lat-**derivations** are finite trees labelled with inequations; e.g.

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Equivalence of $\vdash_{\mathrm{Lat}}$ and $\vdash_{\mathcal{LAT}}$

Theorem

$\vdash_{\mathrm{Lat}}$ and $\vdash_{\mathcal{LAT}}$ are equivalent with transformers defined by

$$\begin{aligned} \tau(\varphi \approx \psi) &= \{\varphi \le \psi, \psi \le \varphi\} \\ \rho(\varphi \le \psi) &= \{\varphi \wedge \psi \approx \varphi\}. \end{aligned}$$

Proof Sketch

It suffices to show that for any set of inequations $\Sigma \cup \{\varphi \leq \psi\}$:

$$\Sigma \vdash_{\mathrm{Lat}} \varphi \leq \psi \iff \Sigma \vdash_{\mathcal{LAT}} \varphi \leq \psi.$$

($\Rightarrow$) By induction on the height of a derivation in Lat.

($\Leftarrow$) We define a binary relation Θ on the lattice formulas Fm by

$$(\varphi, \psi) \in \Theta \quad \Leftrightarrow \quad (\Sigma \vdash_{\mathrm{Lat}} \varphi \leq \psi \quad \text{and} \quad \Sigma \vdash_{\mathrm{Lat}} \psi \leq \varphi).$$

Θ is *reflexive* by (ID), *symmetric* by definition, and *transitive* by (CUT), i.e., an *equivalence relation*. In fact, Θ is a *congruence* on **Fm**. E.g., if $(\varphi_1, \psi_1) \in \Theta$ and $(\varphi_2, \psi_2) \in \Theta$, then $(\varphi_1 \wedge \varphi_2, \psi_1 \wedge \psi_2) \in \Theta$ using

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Moreover, $\mathbf{Fm}/\Theta$ is a lattice. E.g., commutativity of $\wedge^{\mathbf{Fm}/\Theta}$ follows from

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$$\begin{aligned} \varphi/\Theta \le^{\mathbf{Fm}/\Theta} \psi/\Theta \quad &\Leftrightarrow \quad \varphi/\Theta \wedge^{\mathbf{Fm}/\Theta} \psi/\Theta = (\varphi \wedge \psi)/\Theta = \varphi/\Theta \\ &\Leftrightarrow \quad \Sigma \vdash_{\mathrm{Lat}} \varphi \wedge \psi \le \varphi \quad \text{and} \quad \Sigma \vdash_{\mathrm{Lat}} \varphi \le \varphi \wedge \psi \\ &\Leftrightarrow \quad \Sigma \vdash_{\mathrm{Lat}} \varphi \le \psi. \end{aligned}$$

So if $\Sigma \nvdash_{\mathrm{Lat}} \varphi \le \psi$, then $\Sigma \nvdash_{\mathcal{LAT}} \varphi \le \psi$. □

Proof Sketch

Moreover, $\mathbf{Fm}/\Theta$ is a lattice. E.g., commutativity of $\wedge^{\mathbf{Fm}/\Theta}$ follows from

$$\dfrac{\dfrac{\dfrac{}{\psi \leq \psi}\,(\text{ID})}{\varphi \wedge \psi \leq \psi}\,(\wedge\Rightarrow)_2 \qquad \dfrac{\dfrac{}{\varphi \leq \varphi}\,(\text{ID})}{\varphi \wedge \psi \leq \varphi}\,(\wedge\Rightarrow)_1}{\varphi \wedge \psi \leq \psi \wedge \varphi}\,(\Rightarrow\wedge)$$

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Cut-Elimination

Now let Lat° be Lat without (CUT), and let $d \vdash_{\mathrm{Lat}^\circ} \varphi \leq \psi$ denote that d is a derivation of $\varphi \leq \psi$ in Lat°.

Theorem

If $\vdash_{\mathrm{Lat}} \varphi \leq \psi$, *then* $\vdash_{\mathrm{Lat}^\circ} \varphi \leq \psi$.

Proof sketch. Applications of (CUT) can be *eliminated* from a Lat-derivation of $\varphi \leq \psi$ by pushing them upwards until they vanish...

Induction hypothesis. We show that

$$(d_1 \vdash_{\mathrm{Lat}^\circ} \varphi \leq \chi \quad \text{and} \quad d_2 \vdash_{\mathrm{Lat}^\circ} \chi \leq \psi) \quad \Longrightarrow \quad \vdash_{\mathrm{Lat}^\circ} \varphi \leq \psi$$

by induction on the sum of the heights of the derivations d_1 and d_2.

Base case. If d_1 ends with (ID), then $\chi = \varphi$ and d_2 is the required Lat°-derivation of $\varphi \leq \psi$ (similarly, if d_2 ends with (ID)).

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Cut-Elimination

Inductive step. There are two cases:

(1) The "cut-formula" χ is decomposed in both premises, e.g.

$$\dfrac{\dfrac{\dfrac{\vdots}{\varphi \leq \chi_1} \quad \dfrac{\vdots}{\varphi \leq \chi_2}}{\varphi \leq \chi_1 \wedge \chi_2}\,(\Rightarrow\wedge) \qquad \dfrac{\dfrac{\vdots}{\chi_1 \leq \psi}}{\chi_1 \wedge \chi_2 \leq \psi}\,(\wedge\Rightarrow)_1}{\varphi \leq \psi}\,(\textsc{cut}) \quad \Longrightarrow \quad \dfrac{\dfrac{\vdots}{\varphi \leq \chi_1} \quad \dfrac{\vdots}{\chi_1 \leq \psi}}{\varphi \leq \psi}\,(\textsc{cut})$$

(2) The "cut-formula" χ is not decomposed in one premise, e.g.

$$\dfrac{\dfrac{\dfrac{\vdots}{\varphi_1 \leq \chi} \quad \dfrac{\vdots}{\varphi_2 \leq \chi}}{\varphi_1 \vee \varphi_2 \leq \chi}\,(\vee\Rightarrow) \qquad \dfrac{\vdots}{\chi \leq \psi}}{\varphi_1 \vee \varphi_2 \leq \psi}\,(\textsc{cut}) \quad \Longrightarrow \quad \dfrac{\dfrac{\dfrac{\vdots}{\varphi_1 \leq \chi} \quad \dfrac{\vdots}{\chi \leq \psi}}{\varphi_1 \leq \psi}\,(\textsc{cut}) \qquad \dfrac{\dfrac{\vdots}{\varphi_2 \leq \chi} \quad \dfrac{\vdots}{\chi \leq \psi}}{\varphi_2 \leq \psi}\,(\textsc{cut})}{\varphi_1 \vee \varphi_2 \leq \psi}\,(\vee\Rightarrow)$$

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$\square$

Corollaries of Cut-Elimination

Corollary

The equational theory of lattices is decidable.

We also obtain results for free lattices, e.g., Whitman's condition

$$\vdash_{\mathcal{LAT}} \varphi_1 \land \varphi_2 \leq \psi_1 \lor \psi_2 \quad \Longrightarrow \quad \vdash_{\mathcal{LAT}} \varphi_1 \leq \psi_1 \lor \psi_2, \ \vdash_{\mathcal{LAT}} \varphi_2 \leq \psi_1 \lor \psi_2$$

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Part II

Substructural Logics & Residuated Lattices

An Interesting Case Study

We consider...

- on the logical side, **substructural logics**,
- on the algebraic side, **residuated lattices**,
- and the mutually beneficial relationship between the two.

Substructural Logics

Substructural logics are logics that in some sense – they defy precise definition – live "beneath the surface" of classical logic.

Motivated by considerations from *linguistics*, *algebra*, *set theory*, *philosophy*, and *computer science*, they all reject at least one classically valid "structural rule".

(The expression "substructural logic" was proposed by Kosta Došen and Peter Schroeder-Heister at a conference in Tübingen in 1990.)

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Sequents

Let us begin with a proof-theoretic presentation of **classical logic** in a language with connectives $\wedge$, $\vee$, $\to$, 1, and 0, defining $\neg\varphi = \varphi \to 0$.

A **sequent** will be an ordered pair of sequences of formulas, written

$$\varphi_1, \ldots, \varphi_n \ \Rightarrow \ \psi_1, \ldots, \psi_m$$

and possibly understood as

"if all of $\varphi_1, \ldots, \varphi_n$ are true, then one of $\psi_1, \ldots, \psi_m$ is true".

We call such a sequent **single-conclusion** when $m \leq 1$.

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A Sequent Calculus GCL

Initial sequents

$$\dfrac{}{\varphi \Rightarrow \varphi}\ (\text{ID})$$

Cut rule

$$\dfrac{\Gamma_2 \Rightarrow \varphi, \Delta_2 \quad \Gamma_1, \varphi, \Gamma_3 \Rightarrow \Delta_1}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta_1, \Delta_2}\ (\text{CUT})$$

Left structural rules

$$\dfrac{\Gamma_1, \varphi, \psi, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \psi, \varphi, \Gamma_2 \Rightarrow \Delta}\ (\text{EL})$$

$$\dfrac{\Gamma_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \varphi, \Gamma_2 \Rightarrow \Delta}\ (\text{WL})$$

$$\dfrac{\Gamma_1, \varphi, \varphi, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \varphi, \Gamma_2 \Rightarrow \Delta}\ (\text{CL})$$

Right structural rules

$$\dfrac{\Gamma \Rightarrow \Delta_1, \varphi, \psi, \Delta_2}{\Gamma \Rightarrow \Delta_1, \psi, \varphi, \Delta_2}\ (\text{ER})$$

$$\dfrac{\Gamma \Rightarrow \Delta_1, \Delta_2}{\Gamma \Rightarrow \Delta_1, \varphi, \Delta_2}\ (\text{WR})$$

$$\dfrac{\Gamma \Rightarrow \Delta_1, \varphi, \varphi, \Delta_2}{\Gamma \Rightarrow \Delta_1, \varphi, \Delta_2}\ (\text{CR})$$

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A Sequent Calculus GCL

Left operational rules

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, 1, \Gamma_2 \Rightarrow \Delta}\ (1 \Rightarrow) \qquad \frac{}{0 \Rightarrow}\ (0 \Rightarrow)$$

$$\frac{\Gamma_1, \varphi_i, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \varphi_1 \wedge \varphi_2, \Gamma_2 \Rightarrow \Delta}\ (\wedge\Rightarrow)_i \quad i = 1, 2$$

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$$\frac{\Gamma \Rightarrow \Delta_1, \varphi_1, \Delta_2 \quad \Gamma \Rightarrow \Delta_1, \varphi_2, \Delta_2}{\Gamma \Rightarrow \Delta_1, \varphi_1 \wedge \varphi_2, \Delta_2} \ (\Rightarrow\wedge)$$

$$\frac{\Gamma \Rightarrow \Delta_1, \varphi_i, \Delta_2}{\Gamma \Rightarrow \Delta_1, \varphi_1 \vee \varphi_2, \Delta_2} \ (\Rightarrow\vee)_i \quad i = 1, 2$$

$$\frac{\varphi, \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta} \ (\Rightarrow\to)$$

A Sequent Calculus GCL

Left operational rules

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, 1, \Gamma_2 \Rightarrow \Delta}\ (1 \Rightarrow) \qquad \frac{}{0 \Rightarrow}\ (0 \Rightarrow)$$

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$$\frac{\Gamma_2 \Rightarrow \varphi, \Delta_2 \quad \Gamma_1, \psi, \Gamma_3 \Rightarrow \Delta_1}{\Gamma_1, \Gamma_2, \varphi \to \psi, \Gamma_3 \Rightarrow \Delta_1, \Delta_2}\ (\to\Rightarrow)$$

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Peirce's Law

$$\dfrac{\dfrac{(\varphi \to \psi) \to \varphi \Rightarrow \varphi, \varphi}{(\varphi \to \psi) \to \varphi \Rightarrow \varphi}\ (\text{CR})}{\Rightarrow ((\varphi \to \psi) \to \varphi) \to \varphi}\ ({\Rightarrow}{\to})$$

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\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(\mathrm{ID}) \qquad \dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(\mathrm{ID})}{\varphi \Rightarrow \psi, \varphi}\,(\mathrm{WR})}{\Rightarrow \varphi \to \psi, \varphi}\,(\Rightarrow\to)}{(\varphi \to \psi) \to \varphi \Rightarrow \varphi, \varphi}\,(\to\Rightarrow)}{(\varphi \to \psi) \to \varphi \Rightarrow \varphi}\,(\mathrm{CR})}{\Rightarrow ((\varphi \to \psi) \to \varphi) \to \varphi}\,(\Rightarrow\to)
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A First Substructural Logic?

A sequent calculus GIL for **intuitionistic logic** is obtained by restricting the rules of GCL to single-conclusion sequents.

For example, we obtain implication rules

$$\frac{\Gamma_2 \Rightarrow \varphi \quad \Gamma_1, \psi, \Gamma_3 \Rightarrow \Delta}{\Gamma_1, \Gamma_2, \varphi \to \psi, \Gamma_3 \Rightarrow \Delta} \ (\to\Rightarrow) \qquad \frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi} \ (\Rightarrow\to)$$

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Two Conjunctions?

If “I have €1 and I have €1”, how many Euros do I have? €1 or €2?

The rules for $\wedge$ in GCL correspond to the first “additive” meaning of conjunction, but we could also define a “multiplicative” conjunction by

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These are inter-derivable in LJ and LK, e.g.

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Dropping Weakening

Some substructural logics may be obtained by dropping structural rules from GCL and GIL and adding rules for splitting connectives.

E.g., **relevant logics** denying "paradoxes of strict implication" like

$$\varphi \to (\psi \vee \neg\psi) \qquad \text{and} \qquad (\varphi \wedge \neg\varphi) \to \psi$$

are obtained by removing the weakening rules and adding rules for $\cdot$.

The most famous relevant logic R (which admits distributivity) requires a more complicated sequent framework, however.

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Dropping Contraction

In the 1920's, Jan Łukasiewicz introduced logics with $n \geq 3$ truth values and an infinite-valued logic with truth values in $[0, 1]$, where negation and implication are interpreted by the truth functions

$$\neg x = 1 - x \qquad \text{and} \qquad x \to y = \min(1, 1 - x + y).$$

Contraction fails, since the following formula is not valid (constantly 1):

$$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi).$$

These and related contraction-free logics have been used to model vagueness and to (try to) avoid set-theoretic paradoxes.

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Dropping Weakening and Contraction

Other "resource-based" substructural logics are obtained by dropping both weakening and contraction rules.

Girard's **linear logic** also adds rules for the special connectives ! "of course" and ? "why not" that recover structural properties, e.g.

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Dropping all Structural Rules

Lambek's calculus for **grammatical types** has "division operators" $\backslash$ and $/$, where, e.g., the intransitive verb "works" has type $n\backslash s$ and the adjective "poor" has type n/n (with n = noun phrase and s = sentence).

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Extensions of FL

Typically, FL extended appropriately with exchange (e), weakening (w), and contraction (c) rules is denoted by FL_S for $S \subseteq \{e, w, c\}$.

However, we are not limited to these structural rules; consider, e.g.

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \ (\text{MIX}) \qquad \frac{\Gamma, \Gamma \Rightarrow \Delta, \Delta}{\Gamma \Rightarrow \Delta} \ (\text{GC})$$

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From Consequence to Algebra?

Each sequent calculus C gives rise to a **consequence relation** $\vdash_{\mathrm{C}}$ over the set of all sequents of the language.

In particular, for a set of sequents $\Theta \cup \{S\}$:

$$\Theta \vdash_{\mathrm{FL}} S \qquad \Longleftrightarrow \qquad S \text{ is derivable from } \Theta \text{ in FL.}$$

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Residuated Lattices

A **residuated lattice** is an algebra

$$\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$$

such that

- $(A, \wedge, \vee)$ is a lattice
- $(A, \cdot, 1)$ is a monoid
- and for all $x, y, z \in A$

$$x \le z/y \quad \Leftrightarrow \quad x \cdot y \le z \quad \Leftrightarrow \quad y \le x\backslash z.$$

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$$x \leq z/y \quad \Leftrightarrow \quad x \cdot y \leq z \quad \Leftrightarrow \quad y \leq x\backslash z.$$

An **FL-algebra** is a residuated lattice with an extra nullary operation 0. We can also regard a residuated lattice as an FL-algebra with $0 = 1$.

Residuated Lattices

A **residuated lattice** is an algebra

$$\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$$

such that

- $(A, \wedge, \vee)$ is a lattice
- $(A, \cdot, 1)$ is a monoid
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The Commutative Setting

Commutative FL-algebras satisfy the equation

$$x \cdot y \approx y \cdot x$$

and therefore also

$$x \backslash y \approx y / x.$$

So in this case, we just write $\to$ for either $\backslash$ or $/$.

Consider, for example:

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Examples from Rings

Let **R** be a unital ring and $\mathrm{I}(R)$ the lattice of two-sided ideals of **R**.

Consider the operations for $I, J \in \mathrm{I}(R)$:

$$\begin{aligned} I \cdot J &= \{\sum_{k=1}^{n} a_k b_k \mid a_k \in I; b_k \in J; n \geq 1\} \\ I \backslash J &= \{x \in R \mid Ix \subseteq J\} \\ J / I &= \{x \in R \mid xI \subseteq J\}. \end{aligned}$$

Then we obtain an FL-algebra:

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An Equational Axiomatization

The class $\mathcal{FL}$ of FL-algebras is a **variety** defined by the equations for lattices and monoids together with

$$x \cdot (y \vee z) \approx (x \cdot y) \vee (x \cdot z) \qquad (y \vee z) \cdot x \approx (y \cdot x) \vee (z \cdot x)$$

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Significant Classes

Up to term-equivalence...

- **Heyting algebras** are commutative FL-algebras satisfying

$$x \cdot y \approx x \wedge y \qquad \text{and} \qquad 0 \leq x.$$

- **Boolean algebras** are Heyting algebras satisfying ($\neg x = x \to 0$)

$$\neg\neg x \approx x.$$

- **Lattice-ordered groups** are residuated lattices satisfying

$$x \cdot (1/x) \approx 1.$$

- **MV-algebras** are commutative FL-algebras satisfying

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Equivalence of $\vdash_{\mathrm{FL}}$ and $\vdash_{\mathcal{FL}}$

Theorem

$\vdash_{\mathrm{FL}}$ and $\vdash_{\mathcal{FL}}$ are equivalent with transformers defined by

$$\begin{aligned}\tau(\varphi \approx \psi) &= \{\varphi \Rightarrow \psi, \psi \Rightarrow \varphi\}\\ \rho(\varphi_1, \ldots, \varphi_n \Rightarrow \psi) &= \{\varphi_1 \cdot \ldots \cdot \varphi_n \leq \psi\}\\ \rho(\varphi_1, \ldots, \varphi_n \Rightarrow\ ) &= \{\varphi_1 \cdot \ldots \cdot \varphi_n \leq 0\}\end{aligned}$$

where $\varphi_1 \cdot \ldots \cdot \varphi_n$ is 1 *when $n = 0$.*

Decidability

For a given class of FL-algebras $\mathcal{K}$, we might ask...

- is the **equational theory** of $\mathcal{K}$ decidable?

  ($\vdash_{\mathcal{K}} \varphi \approx \psi$ for a given $\mathcal{L}$-equation $\varphi \approx \psi$?)

- is the **quasiequational theory** of $\mathcal{K}$ decidable?

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Cut Elimination

- Decidability of the equational theory of $\mathcal{FL}$ follows immediately (as in the case of lattices) from a proof of cut elimination for FL.
- Decidability follows similarly – but not always immediately – for other varieties of FL-algebras.
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The (Strong) Finite Model Property

A class $\mathcal{K}$ of $\mathcal{L}$-algebras has the **finite model property** (FMP) if

$$\not\models_{\mathcal{K}} \varphi \approx \psi \quad \Longrightarrow \quad \not\models_{\mathbf{A}} \varphi \approx \psi \quad \text{for some finite } \mathbf{A} \in \mathcal{K}$$

and the **strong finite model property** (SFMP) if (for Σ finite)

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Lemma

If $\mathcal{K}$ is **finitely axiomatizable**, *then*

$$\text{FMP} \quad \Longrightarrow \quad \textit{the equational theory of } \mathcal{K} \textit{ is decidable}$$

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Finite Embeddability Property

$\mathcal{K}$ has the **finite embeddability property** (FEP) if

B is a finite partial subalgebra of **A** $\in \mathcal{K}$ $\Longrightarrow$ **B** embeds into some finite **C** $\in \mathcal{K}$.

Lemma

If $\mathcal{K}$ has the FEP, *then $\mathcal{K}$ has the* SFMP *and if it is also finitely axiomatizable, its quasiequational theory is decidable.*

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Lemma

If $\mathcal{K}$ has the FEP, *then $\mathcal{K}$ has the* SFMP *and if it is also finitely axiomatizable, its quasiequational theory is decidable.*

Establishing the FEP

Theorem (McKinsey and Tarski)

The variety $\mathcal{HA}$ of Heyting algebras has the FEP.

Proof.

Let **B** be a finite partial subalgebra of some $\mathbf{A} \in \mathcal{HA}$. Then the lattice **D** generated by $B \cup \{0, 1\}$ is a finitely generated distributive lattice and hence finite. Since the $\wedge$ in any finite distributive lattice is residuated, **D** can be viewed as a Heyting algebra. Moreover, the partially defined residuum operation of **B** coincides (where defined) with the residuum of the meet of **D**, so **B** can be embedded into this algebra. □

More complicated constructions have been introduced by Blok and Van Alten that establish the FEP for many classes of FL-algebras.

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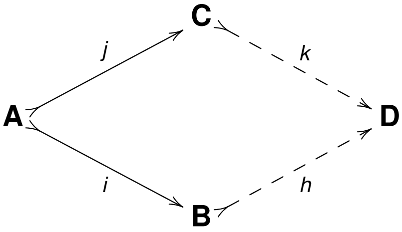
The Amalgamation Property

A class of $\mathcal{L}$-algebras $\mathcal{K}$ has the **amalgamation property** (AP) if for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings i and j of $\mathbf{A}$ into $\mathbf{B}$ and $\mathbf{C}$, there exist $\mathbf{D} \in \mathcal{K}$ and embeddings h, k of $\mathbf{B}$ and $\mathbf{C}$ into $\mathbf{D}$ such that $h \circ i = k \circ j$.



The Deductive Interpolation Property

$\mathcal{K}$ has the **deductive interpolation property** (DIP) if whenever

$$\Sigma \vdash_{\mathcal{K}} \varphi \approx \psi$$

there exists a set of equations Δ satisfying

- $\mathrm{Var}(\Delta) \subseteq \mathrm{Var}(\Sigma) \cap \mathrm{Var}(\varphi \approx \psi)$ ($\mathrm{Var}(X)$ denotes the variables of X)
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Theorem

A variety of commutative FL-algebras has the AP *iff it has the* DIP.

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Proof.

It suffices to show that $\mathcal{HA}$ has the DIP using the calculus GIL. Namely, we can prove that whenever

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Final Remarks

- Relationships between logic and algebra, developed via consequence relations, can be fruitful on both sides.

- A current hot topic is the question of when “good” proof systems for logics / classes of algebras exist, and what role duality and relational semantics play in all of this.

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Further Reading