# LP and its Some of its Metatheory

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# Plan

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- Connectives: $\wedge, \vee, \neg$ ($A \supset B$ can be defined as $\neg A \vee B$)
- Quantifiers: $\forall, \exists$

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- $\delta(f_n)$ is a function from $D^n$ to $D$
- $\delta(P_n)$ is a pair, $\langle \delta^+(P_n), \delta^-(P_n) \rangle$ such that:
    - $\delta^+(P_n) \cup \delta^-(P_n) = D^n$
    - $\delta^+(=) = \{\langle d, d \rangle : d \in D\}$

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- $\mathfrak{A} \Vdash^- \neg A$ iff $\mathfrak{A} \Vdash^+ A$
- $\mathfrak{A} \Vdash^+ A \wedge B$ iff $\mathfrak{A} \Vdash^+ A$ and $\mathfrak{A} \Vdash^+ B$
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- $\Sigma \models_{LP} A$ iff every model of $\Sigma$ is a model of $A$

# Properties of Consequence



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- $\Sigma \models_{CL} A \not\Rightarrow \Sigma \models_{LP} A$
- But $\emptyset \models_{CL} A \Leftrightarrow \emptyset \models_{LP} A$

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- which is a congruence relation on the interpretations of the function symbols in the language:
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- If $d \in D$, $[d]$ is the equivalence class of $d$.

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$$\begin{array}{ccccccccccccc} & & & & & & n+p-1 & \leftarrow & \ldots & \leftarrow & n+3 \\ & & & & & & \downarrow & & & & \uparrow \\ 0 & \rightarrow & 1 & \rightarrow & \ldots & \rightarrow & n & \rightarrow & n+1 & \rightarrow & n+2 \end{array}$$

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- So it is axiomatic