Remarks on Gödel's Incomplentess Theorems

SATO Kentaro*

sato@inf.unibe.ch

SGSLPS Autumn 2016

*His research is supported by John Templeton Foundation

Quiz 1

- Gödel's Completeness Theorem (1929):
  The first order classical logic is complete.

Quiz 1

- Gödel's Completeness Theorem (1929):

  The first order classical logic is complete.

  Henkin (1947) strengthened:

  Any theory over the first order classical logic is complete.

Quiz 1

- Gödel's Completeness Theorem (1929):

  The first order classical logic is complete.

  Henkin (1947) strengthened:

  Any theory over the first order classical logic is complete.

  In particular:

  Peano Arithmetic **PA** *is complete.*

Quiz 1

- Gödel's Completeness Theorem (1929):

  The first order classical logic is complete.

  Henkin (1947) strengthened:

  Any theory over the first order classical logic is complete.

  In particular:

  Peano Arithmetic **PA** *is complete.*

- Gödel's 1st Incompleteness Theorem (1931):

  PA *is incomplete.*

Quiz 1 — Which is correct?

- Gödel's Completeness Theorem (1929):

  The first order classical logic is complete.

  Henkin (1947) strengthened:

  Any theory over the first order classical logic is complete.

  In particular:

  Peano Arithmetic **PA** *is complete.*

- Gödel's 1st Incompleteness Theorem (1931):

  PA *is incomplete.*

Quiz 2

- Hilbert's Programme looks for:

  a complete and decidable axiomatization of real numbers.

Quiz 2

- Hilbert's Programme looks for:

  a complete and decidable axiomatization of real numbers.

  Gödel Incompleteness Theorem answers:

  "*impossible*".

Quiz 2

- Hilbert's Programme looks for:

  a complete and decidable axiomatization of real numbers.

  Gödel Incompleteness Theorem answers:

  "*impossible*".

- Tarski's Theorem (1951):

  quantifier elimination of real closed field.

Quiz 2

- Hilbert's Programme looks for:

  a complete and decidable axiomatization of real numbers.

  Gödel Incompleteness Theorem answers:

  "*impossible*".

- Tarski's Theorem (1951):

  quantifier elimination of real closed field.

  As a consequence, it yields:

  a complete and decidable axiomatization of $(\mathbb{R}, 0, 1, -, +, \cdot, <)$.

Quiz 2 — Which is correct?

- Hilbert's Programme looks for:

  a complete and decidable axiomatization of real numbers.

  Gödel Incompleteness Theorem answers:

  "*impossible*".

- Tarski's Theorem (1951):

  quantifier elimination of real closed field.

  As a consequence, it yields:

  a complete and decidable axiomatization of $(\mathbb{R}, 0, 1, -, +, \cdot, <)$.

Quiz 3

- Gödel 2nd Incompleteness (1931):
  PA *cannot prove a sentence which represents the consistency of* **PA**.

Quiz 3

- Gödel 2nd Incompleteness (1931):
  PA *cannot prove a sentence which represents the consistency of* **PA**.

- Kreisel's Remark (1960):
  PA *does prove a sentence which represents the consistency of* **PA**.

Quiz 3 — Which is correct?

- Gödel 2nd Incompleteness (1931):

  PA *cannot prove a sentence which represents the consistency of* **PA**.

- Kreisel's Remark (1960):

  PA *does prove a sentence which represents the consistency of* **PA**.

Outline

1. Know the statement correctly (40min):
    - the notions in the statement correctly;
    - the preconditions correctly;
    - counterexamples to preconditions?

Outline

1. Know the statement correctly (40min):
   - the notions in the statement correctly;
   - the preconditions correctly;
   - counterexamples to preconditions?
2. A brief look at the proofs (15min):
   - ω-consistency;
   - Gödel sentence vs. Rosser sentence;
   - Kreisel's remark;
   - Loeb's derivability conditions;

Outline

1. Know the statement correctly (40min):
   - the notions in the statement correctly;
   - the preconditions correctly;
   - counterexamples to preconditions?
2. A brief look at the proofs (15min):
   - ω-consistency;
   - Gödel sentence vs. Rosser sentence;
   - Kreisel's remark;
   - Loeb's derivability conditions;
3. Connection to the present-day researches (5min):
   - Gödel hierarchy;
   - my own contributions.

1. Know the Statement Correctly

The Statement

If a first order theory T satisfies the following:

- ...
- ...
- ...

then the following hold:

1st incompleteness: T is incomplete;

2nd incompleteness: T cannot prove a sentence which represents the consistency of T.

The Statement

If a first order theory T satisfies the following:

- ...
- ...
- ...

then the following hold:

1st incompleteness: T is incomplete;

The Statement

If a first order theory T satisfies the following:

- ...
- ...
- ...

then the following hold:

1st incompleteness: T is not complete.

Three Completenesses

Semantical Completeness

"provable" $\Leftrightarrow$ "true in any model":

- (Weak) $\vdash \varphi \iff \models \varphi$;
- (Strong) $\Gamma \vdash \varphi \iff \Gamma \models \varphi$.

Negation Completeness

Arithmetical Completeness

Three Completenesses

Semantical Completeness

"provable" $\Leftrightarrow$ "true in any model":

- (Weak) $\vdash \varphi \iff \models \varphi$;
- (Strong) $\Gamma \vdash \varphi \iff \Gamma \models \varphi$.

Negation Completeness

T can prove or disprove any sentence in L_T:

- $T \vdash \varphi$ or $T \vdash \neg\varphi$ for any sentence $\varphi \in L_T$.

Arithmetical Completeness

Three Completenesses

Semantical Completeness

"provable" $\Leftrightarrow$ "true in any model":

- (Weak) $\vdash \varphi \iff \models \varphi$;
- (Strong) $\Gamma \vdash \varphi \iff \Gamma \models \varphi$.

Negation Completeness

T can prove or disprove any sentence in L_T:

- $T \vdash \varphi$ or $T \vdash \neg\varphi$ for any sentence $\varphi \in L_T$.

Arithmetical Completeness

"provable" $\Leftrightarrow$ "true in the intended model":

- $T \vdash \varphi \iff \mathbb{N} \models \varphi$.

Three Completenesses

Semantical Completeness

- Gödel-Henkin's completeness theorem;
- Kripke completeness
  (modal logics, intuitionistic logic).

Negation Completeness

Arithmetical Completeness

Three Completenesses

Semantical Completeness

- Gödel-Henkin's completeness theorem;
- Kripke completeness
  (modal logics, intuitionistic logic).

Negation Completeness

- Gödel(-Rosser)'s 1st incompleteness theorem;
- completeness of theories of
  algebraic closed / real closed fields

Arithmetical Completeness

Three Completenesses

Semantical Completeness

- Gödel-Henkin's completeness theorem;
- Kripke completeness (modal logics, intuitionistic logic).

Negation Completeness

- Gödel(-Rosser)'s 1st incompleteness theorem;
- completeness of theories of algebraic closed / real closed fields

Arithmetical Completeness

Σ^0_1 completeness (of **Q**, **PA**, **ZFC**, etc.)

Quiz 1 — Which is correct?

- Gödel's Completeness Theorem (1929):

  The first order classical logic is complete.

  Henkin (1947) strengthened:

  Any theory over the first order classical logic is complete.

  In particular:

  Peano Arithmetic **PA** *is complete.*

- Gödel's 1st Incompleteness Theorem (1931):

  PA *is incomplete.*

The Statement (2)

If a first order theory T satisfies the following:

- ...
- ...
- ...

then the following hold:

1st incompleteness: T is not complete.

The Statement (2)

If a first order theory T satisfies the following:

- T is consistent;
- ...
- ...

then the following hold:

1st incompleteness: T is not complete.

The Statement (2)

If a first order theory T satisfies the following:

- T is consistent;
- T is recursively axiomatizable;
- ...

then the following hold:

1st incompleteness: T is not complete.

The Statement (2)

If a first order theory T satisfies the following:

- T is consistent;
- T is recursively axiomatizable;
- T essentially contains Robinson Arithmetic **Q**,

then the following hold:

1st incompleteness: T is not complete.

Consistency

A first order theory T is *consistent* iff

- $T \nvdash \bot$

  and/or

- $T \nvdash \varphi$ for some φ

  and/or

- either $T \nvdash \varphi$ or $T \nvdash \neg\varphi$ for any φ, i.e.,

Consistency

A first order theory T is *consistent* iff

- $T \nvdash \bot$

  and/or

- $T \nvdash \varphi$ for some φ

  and/or

- either $T \nvdash \varphi$ or $T \nvdash \neg\varphi$ for any φ, i.e., it's not the case that $T \vdash \varphi$ and $T \vdash \neg\varphi$.

Consistency

A first order theory T is *consistent* iff

- $T \nvdash \bot$

  and/or

- $T \nvdash \varphi$ for some φ

  and/or

- either $T \nvdash \varphi$ or $T \nvdash \neg\varphi$ for any φ, i.e., it's not the case that $T \vdash \varphi$ and $T \vdash \neg\varphi$.

If T is *not* consistent,

Consistency

A first order theory T is *consistent* iff

- $T \nvdash \bot$

  and/or

- $T \nvdash \varphi$ for some φ

  and/or

- either $T \nvdash \varphi$ or $T \nvdash \neg\varphi$ for any φ, i.e., it's not the case that $T \vdash \varphi$ and $T \vdash \neg\varphi$.

If T is *not* consistent,

- $T \vdash \varphi$ for any φ;
- hence either $T \vdash \varphi$ or $T \vdash \neg\varphi$ (negation completeness).

The Statement (3)

If a first order theory T satisfies the following:

- T is consistent;
- T is recursively axiomatizable;
- T essentially contains Robinson Arithmetic Q,

then the following hold:

1st incompleteness: T is not complete.

Recursive Axiomatizability

A first order theory T is *recursively axiomatizable* iff

- there is Γ such that
  - $\Gamma \vdash \varphi \iff T \vdash \varphi$ for any $\varphi \in L_T$ and
  - $\{\ulcorner\varphi\urcorner \mid \varphi \in \Gamma\}$ is decidable;

Recursive Axiomatizability

A first order theory T is *recursively axiomatizable* iff

- there is Γ such that
  - $\Gamma \vdash \varphi \iff T \vdash \varphi$ for any $\varphi \in L_T$ and
  - $\{\ulcorner\varphi\urcorner \mid \varphi \in \Gamma\}$ is decidable;

  and/or

- there is Γ such that
  - $\Gamma \vdash \varphi \iff T \vdash \varphi$ for any $\varphi \in L_T$ and
  - $\{\ulcorner\varphi\urcorner \mid \varphi \in \Gamma\}$ is semi-decidable;

Recursive Axiomatizability

A first order theory T is *recursively axiomatizable* iff

- there is Γ such that
  - $\Gamma \vdash \varphi \iff T \vdash \varphi$ for any $\varphi \in L_T$ and
  - $\{\ulcorner\varphi\urcorner \mid \varphi \in \Gamma\}$ is decidable;

  and/or
- there is Γ such that
  - $\Gamma \vdash \varphi \iff T \vdash \varphi$ for any $\varphi \in L_T$ and
  - $\{\ulcorner\varphi\urcorner \mid \varphi \in \Gamma\}$ is semi-decidable;

  and/or
- $\{\ulcorner\varphi\urcorner \mid T \vdash \varphi\}$ is semi-decidable.

Recursive Axiomatizability

A first order theory T is *recursively axiomatizable* iff

- there is Γ such that
  - $\Gamma \vdash \varphi \iff T \vdash \varphi$ for any $\varphi \in L_T$ and
  - $\{\ulcorner\varphi\urcorner \mid \varphi \in \Gamma\}$ is decidable;

  and/or
- there is Γ such that
  - $\Gamma \vdash \varphi \iff T \vdash \varphi$ for any $\varphi \in L_T$ and
  - $\{\ulcorner\varphi\urcorner \mid \varphi \in \Gamma\}$ is semi-decidable;

  and/or
- $\{\ulcorner\varphi\urcorner \mid T \vdash \varphi\}$ is semi-decidable.

$\mathrm{Th}(\mathbb{N}) = \{\varphi \in L_{\mathbf{PA}} \mid \mathbb{N} \models \varphi\}$ is negation complete.

Craig's Theorem

If $\{\ulcorner\varphi\urcorner \mid T \vdash \varphi\}$ is semi-decidable,
then there is Γ such that

- $\Gamma \vdash \varphi \iff T \vdash \varphi$ for any $\varphi \in L_T$ and
- $\{\ulcorner\varphi\urcorner \mid \varphi \in \Gamma\}$ is decidable

Craig's Theorem

If $\{\ulcorner\varphi\urcorner \mid T \vdash \varphi\}$ is semi-decidable,
then there is Γ such that

- $\Gamma \vdash \varphi \iff T \vdash \varphi$ for any $\varphi \in L_T$ and
- $\{\ulcorner\varphi\urcorner \mid \varphi \in \Gamma\}$ is decidable

(Proof) Take a recursive predicate R such that

$$T \vdash \varphi \iff \exists n R(\ulcorner\varphi\urcorner, n) \text{ for any } \varphi \in L_T.$$

Craig's Theorem

If $\{\ulcorner\varphi\urcorner \mid T \vdash \varphi\}$ is semi-decidable,
then there is Γ such that

- $\Gamma \vdash \varphi \iff T \vdash \varphi$ for any $\varphi \in L_T$ and
- $\{\ulcorner\varphi\urcorner \mid \varphi \in \Gamma\}$ is decidable

(Proof) Take a recursive predicate R such that

$$T \vdash \varphi \iff \exists n R(\ulcorner\varphi\urcorner, n) \text{ for any } \varphi \in L_T.$$

Define the following recursive set of axioms

$$\Gamma = \{\psi \mid (\exists n, \ulcorner\varphi\urcorner < \ulcorner\psi\urcorner)(R(\ulcorner\varphi\urcorner, n) \,\&\, \psi \equiv \varphi \wedge \ldots \wedge \varphi)\}.$$

Craig's Theorem

If $\{\ulcorner\varphi\urcorner \mid T \vdash \varphi\}$ is semi-decidable,
then there is Γ such that

- $\Gamma \vdash \varphi \iff T \vdash \varphi$ for any $\varphi \in L_T$ and
- $\{\ulcorner\varphi\urcorner \mid \varphi \in \Gamma\}$ is decidable

(Proof) Take a recursive predicate R such that

$$T \vdash \varphi \iff \exists n R(\ulcorner\varphi\urcorner, n) \text{ for any } \varphi \in L_T.$$

Define the following recursive set of axioms

$$\Gamma = \{\psi \mid (\exists n, \ulcorner\varphi\urcorner < \ulcorner\psi\urcorner)(R(\ulcorner\varphi\urcorner, n) \,\&\, \psi \equiv \varphi\wedge\ldots\wedge\varphi)\}.$$

- $\psi \in \Gamma \Rightarrow T \vdash \varphi \,\&\, \psi \equiv \varphi\wedge\ldots\wedge\varphi \Rightarrow T \vdash \psi$;
- $T \vdash \varphi \Rightarrow \exists n R(\ulcorner\varphi\urcorner, n) \Rightarrow \underbrace{\varphi\wedge\ldots\wedge\varphi}_{n+1} \in \Gamma \Rightarrow \Gamma \vdash \varphi.$

Henkin Construction

Henkin's Lemma:
If $\Gamma \nvdash \bot$ then there is maximal consistent $\Delta \supseteq \Gamma$.

Henkin Construction

Henkin's Lemma:
If $\Gamma \nvdash \bot$ then there is maximal consistent $\Delta \supseteq \Gamma$.

(Proof) Let φ_n's enumerate all L formulae. Define

$$\Gamma_{n+1} := \begin{cases} \Gamma_n & \text{if } \Gamma_n \cup \{\varphi_n\} \vdash \bot \\ \Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \nvdash \bot. \end{cases}$$

starting from $\Gamma_0 := \Gamma$. Take $\Delta := \bigcup_{n \in \omega} \Gamma_n$. $\square$

Henkin Construction

Henkin's Lemma:
If $\Gamma \nvdash \bot$ then there is maximal consistent $\Delta \supseteq \Gamma$.

(Proof) Let φ_n's enumerate all L formulae. Define

$$\Gamma_{n+1} := \begin{cases} \Gamma_n & \text{if } \Gamma_n \cup \{\varphi_n\} \vdash \bot \\ \Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \nvdash \bot. \end{cases}$$

starting from $\Gamma_0 := \Gamma$. Take $\Delta := \bigcup_{n \in \omega} \Gamma_n$. $\square$

Note:
The theory generated by Δ is negation complete:
either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$ holds for any $\varphi \in L$.

The Statement (4)

If a first order theory T satisfies the following:

- T is consistent;
- T is recursively axiomatizable;
- T essentially contains Robinson Arithmetic Q,

then the following hold:

1st incompleteness: T is not complete.

Robinson Arithmetic Q

Language (function) 0; S(-); $+$, $\cdot$; (relation) $<$.

Axioms 1. $\neg(S(x) = 0)$;

2. $S(x) = S(y) \to x = y$;
3. $x = 0 \vee \exists y(x = S(y))$;
4. $x+0 = x$; and $x+S(y) = S(x+y)$;
5. $x\cdot 0 = 0$; and $x\cdot S(y) = (x\cdot y)+x$.;
6. $x < y \leftrightarrow \exists z(x+S(z) = y)$.

Robinson Arithmetic Q

Language (function) 0; $S(\text{-})$; $+$, $\cdot$; (relation) $<$.

Axioms 1. $\neg(S(x) = 0)$;

2. $S(x) = S(y) \rightarrow x = y$;
3. $x = 0 \vee \exists y(x = S(y))$;
4. $x + 0 = x$; and $x + S(y) = S(x + y)$;
5. $x \cdot 0 = 0$; and $x \cdot S(y) = (x \cdot y) + x$.;
6. $x < y \leftrightarrow \exists z(x + S(z) = y)$.

Remarks

- first introduced by R. M. Robison in 1950 w/o $<$;
- has no induction axiom (schema).

Theories containing Q

- **PA** extends **Q** by induction scheme:
  $\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(S(x)) \to \forall x \varphi(x)$ for $\varphi \in L_{\mathbf{Q}}$.

Theories containing Q

- **PA** extends **Q** by induction scheme:
  $\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(S(x)) \to \forall x \varphi(x)$ for $\varphi \in L_{\mathbf{Q}}$.
- $\mathbf{I\Sigma}_n$ extends **Q** by induction for Σ^0_n formulae:
  1. $\Sigma^0_n = \{\exists x_n \forall x_{n-1} ... Q x_1 \varphi(\vec{x}) \mid \varphi \in \Delta^0_0\}$ and
  2. $\varphi \in \Delta^0_0$ iff all quantifiers in φ are bounded (i.e., of the forms $\forall x < t$ and $\exists x < t$).

Theories containing Q

- **PA** extends **Q** by induction scheme:

  $\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(S(x)) \to \forall x \varphi(x)$ for $\varphi \in L_{\mathbf{Q}}$.

- $\mathbf{I\Sigma}_n$ extends **Q** by induction for Σ^0_n formulae:
  1. $\Sigma^0_n = \{\exists x_n \forall x_{n-1} ... Q x_1 \varphi(\vec{x}) \mid \varphi \in \Delta^0_0\}$ and
  2. $\varphi \in \Delta^0_0$ iff all quantifiers in φ are bounded (i.e., of the forms $\forall x < t$ and $\exists x < t$).

- **PRA** extends **Q** by
  1. $L_{\mathbf{PRA}} := L_{\mathbf{Q}} \cup \{\underline{\mathcal{F}} \mid \mathcal{F} \in \mathrm{PrimRec}\}$;
  2. induction for *quantifier-free* $L_{\mathbf{PRA}}$ formulae.

Theories containing Q

- **PA** extends **Q** by induction scheme:
  $\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(S(x)) \rightarrow \forall x \varphi(x)$ for $\varphi \in L_{\mathbf{Q}}$.
- $\mathbf{I\Sigma}_n$ extends **Q** by induction for Σ^0_n formulae:
  1. $\Sigma^0_n = \{\exists x_n \forall x_{n-1} ... Q x_1 \varphi(\vec{x}) \mid \varphi \in \Delta^0_0\}$ and
  2. $\varphi \in \Delta^0_0$ iff all quantifiers in φ are bounded (i.e., of the forms $\forall x < t$ and $\exists x < t$).
- **PRA** extends **Q** by
  1. $L_{\mathbf{PRA}} := L_{\mathbf{Q}} \cup \{\underline{\mathcal{F}} \mid \mathcal{F} \in \mathrm{PrimRec}\}$;
  2. induction for *quantifier-free* $L_{\mathbf{PRA}}$ formulae.
- **ZFC** extends **Q** ...

Theories containing Q

- **PA** extends **Q** by induction scheme:
  $\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(S(x)) \rightarrow \forall x \varphi(x)$ for $\varphi \in L_{\mathbf{Q}}$.
- $\mathbf{I\Sigma}_n$ extends **Q** by induction for Σ^0_n formulae:
  1. $\Sigma^0_n = \{\exists x_n \forall x_{n-1} ... Q x_1 \varphi(\vec{x}) \mid \varphi \in \Delta^0_0\}$ and
  2. $\varphi \in \Delta^0_0$ iff all quantifiers in φ are bounded (i.e., of the forms $\forall x < t$ and $\exists x < t$).
- **PRA** extends **Q** by
  1. $L_{\mathbf{PRA}} := L_{\mathbf{Q}} \cup \{\underline{\mathcal{F}} \mid \mathcal{F} \in \mathrm{PrimRec}\}$;
  2. induction for *quantifier-free* $L_{\mathbf{PRA}}$ formulae.
- **ZFC** extends **Q** ... really? in which sense?

Interpretation

An interpretation I of L in L' consists of:

- an L' formula $v_I(x)$, called universe;
- for function $f(\vec{x})$ of L, an L' formula $f^I(y, \vec{x})$;
- for relation $R(\vec{x})$ of L, an L' formula $R^I(\vec{x})$.

Interpretation

An interpretation I of L in L' consists of:

- an L' formula $v_I(x)$, called universe;
- for function $f(\vec{x})$ of L, an L' formula $f^I(y, \vec{x})$;
- for relation $R(\vec{x})$ of L, an L' formula $R^I(\vec{x})$.

Extend I to all L-terms and L-formulae:

- if $t(\vec{x}) \equiv f(t_1(\vec{x}), ..., t_k(\vec{x}))$, then
$t^I(y, \vec{x}) \equiv \exists z_1, ..., z_k(\bigwedge_{i \leq k} {t_i}^I(z_i, \vec{x}) \wedge f^I(y, z_1, .., z_n));$

Interpretation

An interpretation I of L in L' consists of:

- an L' formula $v_I(x)$, called universe;
- for function $f(\vec{x})$ of L, an L' formula $f^I(y, \vec{x})$;
- for relation $R(\vec{x})$ of L, an L' formula $R^I(\vec{x})$.

Extend I to all L-terms and L-formulae:

- if $t(\vec{x}) \equiv f(t_1(\vec{x}), ..., t_k(\vec{x}))$, then $t^I(y, \vec{x}) \equiv \exists z_1, ..., z_k(\bigwedge_{i \leq k} {t_i}^I(z_i, \vec{x}) \wedge f^I(y, z_1, .., z_n))$;
- if $\varphi \equiv R(t_1(\vec{x}), ..., t_k(\vec{x}))$, then $\varphi^I \equiv \exists z_1, ..., z_k(\bigwedge_{i \leq k} {t_i}^I(z_i, \vec{x}) \wedge R^I(z_1, .., z_n))$;

Interpretation

An interpretation I of L in L' consists of:

- an L' formula $v_I(x)$, called universe;
- for function $f(\vec{x})$ of L, an L' formula $f^I(y, \vec{x})$;
- for relation $R(\vec{x})$ of L, an L' formula $R^I(\vec{x})$.

Extend I to all L-terms and L-formulae:

- if $t(\vec{x}) \equiv f(t_1(\vec{x}), ..., t_k(\vec{x}))$, then $t^I(y, \vec{x}) \equiv \exists z_1, ..., z_k (\bigwedge_{i \le k} {t_i}^I(z_i, \vec{x}) \wedge f^I(y, z_1, .., z_n))$;
- if $\varphi \equiv R(t_1(\vec{x}), ..., t_k(\vec{x}))$, then $\varphi^I \equiv \exists z_1, ..., z_k (\bigwedge_{i \le k} {t_i}^I(z_i, \vec{x}) \wedge R^I(z_1, .., z_n))$;
- $(\varphi \wedge \psi)^I \equiv \varphi^I \wedge \psi^I$; and $(\neg\varphi)^I \equiv \neg\varphi^I$;

Interpretation

An interpretation I of L in L' consists of:

- an L' formula $v_I(x)$, called universe;
- for function $f(\vec{x})$ of L, an L' formula $f^I(y, \vec{x})$;
- for relation $R(\vec{x})$ of L, an L' formula $R^I(\vec{x})$.

Extend I to all L-terms and L-formulae:

- if $t(\vec{x}) \equiv f(t_1(\vec{x}), ..., t_k(\vec{x}))$, then $t^I(y, \vec{x}) \equiv \exists z_1, ..., z_k(\bigwedge_{i \leq k} {t_i}^I(z_i, \vec{x}) \wedge f^I(y, z_1, .., z_n))$;
- if $\varphi \equiv R(t_1(\vec{x}), ..., t_k(\vec{x}))$, then $\varphi^I \equiv \exists z_1, ..., z_k(\bigwedge_{i \leq k} {t_i}^I(z_i, \vec{x}) \wedge R^I(z_1, .., z_n))$;
- $(\varphi \wedge \psi)^I \equiv \varphi^I \wedge \psi^I$; and $(\neg\varphi)^I \equiv \neg\varphi^I$;
- $(\forall y \varphi(y))^I \equiv \forall y(v_I(y) \rightarrow \varphi(y)^I)$.

Interpretation 2

Given an interpretation I of L in L'.

- I is an interpretation in an L' theory T' iff
  1. $T' \vdash \exists x \upsilon_I(x)$;
  2. $T' \vdash \forall \vec{x}(\upsilon_I(\vec{x}) \rightarrow \exists ! y(\upsilon_I(y) \wedge f^I(y, \vec{x})))$.

Interpretation 2

Given an interpretation I of L in L'.

- I is an interpretation in an L' theory T' iff
  1. $T' \vdash \exists x v_I(x)$;
  2. $T' \vdash \forall \vec{x}(v_I(\vec{x}) \to \exists! y(v_I(y) \wedge f^I(y, \vec{x})))$.
- I is an interpretation of an L theory T in T' iff
  1. (as above);
  2. (as above);
  3. if $T \vdash \varphi$ then $T' \vdash \varphi^I$ for any $\varphi \in L$.

Theories ess. containing Q

“T' ess. contains T” $=$ “$\exists$ interpretation of T in T'”.

Theories ess. containing Q

"T' ess. contains T" = "$\exists$ interpretation of T in T'".

- **ZFC** essentially contains **Q**
  by von Neumann interpretation **v**:
  1. $v_{\mathbf{v}}(x) \equiv$ "x is a finite von Neumann ordinal";
  2. $S^{\mathbf{v}}(y, x) \equiv y = x \cup \{x\}$, etc.;

Theories ess. containing Q

"T' ess. contains T" = "$\exists$ interpretation of T in T'".

- **ZFC** essentially contains **Q** by von Neumann interpretation **v**:
  1. $v_{\mathbf{v}}(x) \equiv$ "x is a finite von Neumann ordinal";
  2. $S^{\mathbf{v}}(y,x) \equiv y = x \cup \{x\}$, etc.;
- modal extensions of **PA** (directly) contains **Q**;

Theories ess. containing Q

"T' ess. contains T" = "$\exists$ interpretation of T in T'".

- **ZFC** essentially contains **Q** by von Neumann interpretation **v**:
  1. $v_{\mathbf{v}}(x) \equiv$ "x is a finite von Neumann ordinal";
  2. $S^{\mathbf{v}}(y, x) \equiv y = x \cup \{x\}$, etc.;
- modal extensions of **PA** (directly) contains **Q**;
- Heyting Arithmetic **HA** (ess.) contains **Q** by
  - **HA** literally extends **PA** in $\wedge, \neg, \forall$, with extra-operators $\vee, \exists$ (like modality);

Theories ess. containing Q

"T' ess. contains T" = "$\exists$ interpretation of T in T'".

- **ZFC** essentially contains **Q**
  by von Neumann interpretation **v**:
  1. $v_{\mathbf{v}}(x) \equiv$ "x is a finite von Neumann ordinal";
  2. $S^{\mathbf{v}}(y,x) \equiv y = x \cup \{x\}$, etc.;
- modal extensions of **PA** (directly) contains **Q**;
- Heyting Arithmetic **HA** (ess.) contains **Q** by
  - **HA** literally extends **PA** in $\wedge, \neg, \forall$,
    with extra-operators $\vee, \exists$ (like modality);
  - relaxing the notion of interpretation so that
    double negation translation N is included:

$$(\varphi \vee \psi)^N \equiv \neg(\neg\varphi^N \wedge \neg\psi^N);\ (\exists x \varphi(x))^N \equiv \neg\forall x \neg\varphi(x)^N;\ \text{etc.}$$

Presburger Arithmetic PresA

Language $L_{\mathbf{PresA}} = \{0, S, +\}$;

Axioms 1. $\neg(S(x) = 0)$;

2. $S(x) = S(y) \rightarrow x = y$;
3. $x = 0 \vee \exists y(x = S(y))$;
4. $x + 0 = x$; and $x + S(y) = S(x + y)$;
5. induction for all $L_{\mathbf{PresA}}$ formulae.

Presburger Arithmetic PresA

Language $L_{\mathbf{PresA}} = \{0, S, +\}$;

Axioms 1. $\neg(S(x) = 0)$;

2. $S(x) = S(y) \to x = y$;
3. $x = 0 \vee \exists y(x = S(y))$;
4. $x + 0 = x$; and $x + S(y) = S(x + y)$;
5. induction for all $L_{\mathbf{PresA}}$ formulae.

Remarks

- essentially, the $\cdot$-free fragment of PA;

Presburger Arithmetic PresA

Language $L_{\mathbf{PresA}} = \{0, S, +\}$;

Axioms 1. $\neg(S(x) = 0)$;

2. $S(x) = S(y) \rightarrow x = y$;
3. $x = 0 \vee \exists y(x = S(y))$;
4. $x+0 = x$; and $x+S(y) = S(x+y)$;
5. induction for all $L_{\mathbf{PresA}}$ formulae.

Remarks

- essentially, the $\cdot$-free fragment of PA;
- introduced by M. Presburger in 1929;

Presburger Arithmetic PresA

Language $L_{\mathbf{PresA}} = \{0, S, +\}$;

Axioms 1. $\neg(S(x) = 0)$;

2. $S(x) = S(y) \to x = y$;
3. $x = 0 \vee \exists y(x = S(y))$;
4. $x+0 = x$; and $x+S(y) = S(x+y)$;
5. induction for all $L_{\mathbf{PresA}}$ formulae.

Remarks

- essentially, the $\cdot$-free fragment of PA;
- introduced by M. Presburger in 1929;
- proven by him to be complete, i.e.,

  $\mathbf{PreA} \vdash \varphi \iff \mathbb{N} \models \varphi$ for any $\varphi \in L_{\mathbf{PresA}}$;

Presburger Arithmetic PresA

Language $L_{\mathbf{PresA}} = \{0, S, +\}$;

Axioms 1. $\neg(S(x) = 0)$;

2. $S(x) = S(y) \to x = y$;
3. $x = 0 \vee \exists y(x = S(y))$;
4. $x + 0 = x$; and $x + S(y) = S(x + y)$;
5. induction for all $L_{\mathbf{PresA}}$ formulae.

Remarks

- essentially, the $\cdot$-free fragment of **PA**;
- introduced by M. Presburger in 1929;
- proven by him to be complete, i.e.,
  $\mathbf{PreA} \vdash \varphi \iff \mathbb{N} \models \varphi$ for any $\varphi \in L_{\mathbf{PresA}}$;
- hence not essentially contains **Q**.

Theory of real closed fields RCF

Language $L_{\mathbf{RCF}} := \{0, 1, -, +, \cdot, <\}$;

Axioms 1. $x+0=x$; $x+(-x)=0$; $x+y=y+x$;

2. $x\cdot 0=0$; $x\cdot(y+z)=x\cdot y+x\cdot z$; $x\cdot y=y\cdot x$;

3. $x<y \to x+z<y+z$; $x>0 \wedge y>0 \to x\cdot y>0$;

4. $x>0 \to \exists y(x=y\cdot y)$;

5. $\forall x_{2n+1}\ldots x_0(x_{2n+1}\neq 0 \to \exists y(\sum_{i\leq 2n+1} x_i\cdot y^i=0))$.

Theory of real closed fields RCF

Language $L_{\mathbf{RCF}} := \{0, 1, -, +, \cdot, <\}$;

Axioms 1. $x+0=x$; $\quad x+(-x)=0$; $\quad x+y=y+x$;

2. $x\cdot 0=0$; $\quad x\cdot(y+z)=x\cdot y+x\cdot z$; $\quad x\cdot y=y\cdot x$;

3. $x<y \to x+z<y+z$; $\quad x>0 \wedge y>0 \to x\cdot y>0$;

4. $x>0 \to \exists y(x=y\cdot y)$;

5. $\forall x_{2n+1} \ldots x_0 (x_{2n+1} \neq 0 \to \exists y(\sum_{i \leq 2n+1} x_i \cdot y^i = 0))$.

Remarks

- proven by Tarski (1951) to admit quantifier-elimination; and so
- $\mathbf{RFC} \vdash \varphi \iff (\mathbb{R}, 0, 1, -, +, \cdot, <) \models \varphi$;

Theory of real closed fields RCF

Language $L_{\mathbf{RCF}} := \{0, 1, -, +, \cdot, <\}$;

Axioms 1. $x+0=x$; $x+(-x)=0$; $x+y=y+x$;

2. $x\cdot 0=0$; $x\cdot(y+z)=x\cdot y+x\cdot z$; $x\cdot y=y\cdot x$;

3. $x<y \to x+z<y+z$; $x>0 \wedge y>0 \to x\cdot y>0$;

4. $x>0 \to \exists y(x=y\cdot y)$;

5. $\forall x_{2n+1} \ldots x_0 (x_{2n+1} \neq 0 \to \exists y(\sum_{i \leq 2n+1} x_i \cdot y^i = 0))$.

Remarks

- proven by Tarski (1951) to admit quantifier-elimination; and so
- $\mathbf{RFC} \vdash \varphi \iff (\mathbb{R}, 0, 1, -, +, \cdot, <) \models \varphi$;
- hence not essentially contains $\mathbf{Q}$.

Quiz 2 — Which is correct?

- Hilbert's Programme looks for:

  a complete and decidable axiomatization of real numbers.

  Gödel Incompleteness Theorem answers:

  "*impossible*".

- Tarski's Theorem (1951):

  quantifier elimination of real closed field.

  As a consequence, it yields:

  a complete and decidable axiomatization of $(\mathbb{R}, 0, 1, -, +, \cdot, <)$.

The Statement (5)

If a first order theory T satisfies the following:

- T is consistent;
- T is recursively axiomatizable;
- T essentially contains Robinson Arithmetic **Q**,

then the following hold:

1st incompleteness: T is not complete;

The Statement (5)

If a first order theory T satisfies the following:

- T is consistent;
- T is recursively axiomatizable;
- T essentially contains Robinson Arithmetic **Q**,

then the following hold:

1st incompleteness: T is not complete;

2nd incompleteness: T cannot prove a sentence which represents the consistency of T.

Nelson's trick

There is an interpretation of $\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1$ in $\mathbf{Q}$.

- Main idea: define an $L_{\mathbf{Q}}$ formula $W(x)$ which intuitively means "$<$ is well-founded below x";

Nelson's trick

There is an interpretation of $\mathbf{I\Sigma_0 + \Omega_1}$ in $\mathbf{Q}$.

- Main idea: define an $L_{\mathbf{Q}}$ formula $W(x)$ which intuitively means "$<$ is well-founded below x";
- Nelson's interpretation $\mathbf{n}$ should be
  1. $v_{\mathbf{n}}(x) \equiv W(x)$;
  2. $S^{\mathbf{n}}(y,x) \equiv y = S(x)$;
     $+^{\mathbf{n}}(z,x,y) \equiv z = x+y$; $\cdot^{\mathbf{n}}(z,x,y) \equiv z = x \cdot y$.
  3. $=^{\mathbf{n}}(x,y) \equiv x = y$; $\quad <^{\mathbf{n}}(x,y) \equiv x < y$.

Nelson's trick

There is an interpretation of $\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1$ in $\mathbf{Q}$.

- Main idea: define an $L_{\mathbf{Q}}$ formula $W(x)$ which intuitively means "$<$ is well-founded below x";
- Nelson's interpretation $\mathbf{n}$ should be
  1. $v_{\mathbf{n}}(x) \equiv W(x)$;
  2. $S^{\mathbf{n}}(y,x) \equiv y = S(x)$;
     $+^{\mathbf{n}}(z,x,y) \equiv z = x+y$; $\cdot^{\mathbf{n}}(z,x,y) \equiv z = x \cdot y$.
  3. $=^{\mathbf{n}}(x,y) \equiv x = y$; $\quad <^{\mathbf{n}}(x,y) \equiv x < y$.
- Compare to $\varphi \mapsto \varphi^{\mathrm{WF}}$ in set theory.

Nelson's trick

There is an interpretation of $\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1$ in $\mathbf{Q}$.

- Main idea: define an $L_{\mathbf{Q}}$ formula $W(x)$ which intuitively means "$<$ is well-founded below x";
- Nelson's interpretation $\mathbf{n}$ should be
  1. $v_{\mathbf{n}}(x) \equiv W(x)$;
  2. $S^{\mathbf{n}}(y,x) \equiv y = S(x)$;
     $+^{\mathbf{n}}(z,x,y) \equiv z = x+y$; $\cdot^{\mathbf{n}}(z,x,y) \equiv z = x \cdot y$.
  3. $=^{\mathbf{n}}(x,y) \equiv x = y$; $\quad <^{\mathbf{n}}(x,y) \equiv x < y$.
- Compare to $\varphi \mapsto \varphi^{\mathrm{WF}}$ in set theory.

As a consequence,

"T ess. contains $\mathbf{Q}$" $\iff$ "T ess. contains $\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1$"

Numeralwise representation

$R \subseteq \omega^n$ is numeralwise represented by $\varphi(\vec{x})$ iff

- $\mathbf{Q} \vdash \varphi(\overline{k_1}, ..., \overline{k_n}) \iff R(k_1, ..., k_n)$ and
- $\mathbf{Q} \vdash \neg\varphi(\overline{k_1}, ..., \overline{k_n}) \iff \neg R(k_1, ..., k_n)$,

where $\overline{k} := \underbrace{S(...(S}_{k}(0)...)$.

Numeralwise representation

$R \subseteq \omega^n$ is numeralwise represented by $\varphi(\vec{x})$ iff

- $\mathbf{Q} \vdash \varphi(\overline{k_1}, ..., \overline{k_n}) \iff R(k_1, ..., k_n)$ and
- $\mathbf{Q} \vdash \neg\varphi(\overline{k_1}, ..., \overline{k_n}) \iff \neg R(k_1, ..., k_n)$,

where $\overline{k} := \underbrace{S(...(S}_{k}(0)...)$.

We have the following $L_{\mathbf{Q}}$ formulae (Σ^0_1 completeness):

- $\mathbf{Q} \vdash \mathrm{neg}(\overline{\ulcorner\varphi\urcorner}, \overline{k}) \iff k = \ulcorner\neg\varphi\urcorner$ and
  $\mathbf{Q} \vdash \neg\mathrm{neg}(\overline{\ulcorner\varphi\urcorner}, \overline{k}) \iff k \neq \ulcorner\neg\varphi\urcorner$;

Numeralwise representation

$R \subseteq \omega^n$ is numeralwise represented by $\varphi(\vec{x})$ iff

- $\mathbf{Q} \vdash \varphi(\overline{k_1}, ..., \overline{k_n}) \iff R(k_1, ..., k_n)$ and
- $\mathbf{Q} \vdash \neg\varphi(\overline{k_1}, ..., \overline{k_n}) \iff \neg R(k_1, ..., k_n)$,

where $\overline{k} := \underbrace{S(...(S}_{k}(0)...)$.

We have the following $L_{\mathbf{Q}}$ formulae (Σ^0_1 completeness):

- $\mathbf{Q} \vdash \mathrm{neg}(\overline{\ulcorner\varphi\urcorner}, \overline{k}) \iff k = \ulcorner\neg\varphi\urcorner$ and
  $\mathbf{Q} \vdash \neg\mathrm{neg}(\overline{\ulcorner\varphi\urcorner}, \overline{k}) \iff k \neq \ulcorner\neg\varphi\urcorner$;
- $\mathbf{Q} \vdash \mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is a T-proof of φ
  $\mathbf{Q} \vdash \neg\mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is not a T-proof of φ
  (if T is recursively axiomatizable).

Numeralwise representation

$R \subseteq \omega^n$ is numeralwise represented by $\varphi(\vec{x})$ iff

- $\mathbf{Q} \vdash \varphi(\overline{k_1}, \ldots, \overline{k_n}) \iff R(k_1, \ldots, k_n)$ and
- $\mathbf{Q} \vdash \neg\varphi(\overline{k_1}, \ldots, \overline{k_n}) \iff \neg R(k_1, \ldots, k_n)$,

where $\overline{k} := \underbrace{S(\ldots(S}_{k}(0)\ldots)$.

We have the following $L_{\mathbf{Q}}$ formulae (Σ^0_1 completeness):

- $\mathbf{Q} \vdash \mathrm{neg}(\overline{\ulcorner\varphi\urcorner}, \overline{k}) \iff k = \ulcorner\neg\varphi\urcorner$ and
  $\mathbf{Q} \vdash \neg\mathrm{neg}(\overline{\ulcorner\varphi\urcorner}, \overline{k}) \iff k \neq \ulcorner\neg\varphi\urcorner$;
- $\mathbf{Q} \vdash \mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is a T-proof of φ
  $\mathbf{Q} \vdash \neg\mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is not a T-proof of φ
  (if T is recursively axiomatizable).

Then it is natural to define

$$\mathrm{Con}(T) :\equiv \neg\exists x \mathrm{Prf}_T(x, \overline{\ulcorner\bot\urcorner}).$$

Ambiguity

Even if the following hold for all Λ and φ:

- $\mathbf{Q} \vdash \mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is a T-proof of φ and
  $\mathbf{Q} \vdash \neg\mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is not a T-proof of φ;
- $\mathbf{Q} \vdash \mathrm{Prf}^*_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is a T-proof of φ and
  $\mathbf{Q} \vdash \neg\mathrm{Prf}^*_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is not a T-proof of φ;

Ambiguity

Even if the following hold for all Λ and φ:

- $\mathbf{Q} \vdash \mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is a T-proof of φ and
  $\mathbf{Q} \vdash \neg\mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is not a T-proof of φ;
- $\mathbf{Q} \vdash \mathrm{Prf}^*_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is a T-proof of φ and
  $\mathbf{Q} \vdash \neg\mathrm{Prf}^*_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is not a T-proof of φ;

we do *not* have

- $\mathbf{Q} \vdash \forall x, y(\mathrm{Prf}_T(x, y) \leftrightarrow \mathrm{Prf}^*_T(x, y))$, nor

Ambiguity

Even if the following hold for all Λ and φ:

- $\mathbf{Q} \vdash \mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is a T-proof of φ and
  $\mathbf{Q} \vdash \neg\mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is not a T-proof of φ;
- $\mathbf{Q} \vdash \mathrm{Prf}^*_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is a T-proof of φ and
  $\mathbf{Q} \vdash \neg\mathrm{Prf}^*_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is not a T-proof of φ;

we do *not* have

- $\mathbf{Q} \vdash \forall x, y(\mathrm{Prf}_T(x, y) \leftrightarrow \mathrm{Prf}^*_T(x, y))$, nor
- $\mathbf{Q} \vdash \mathrm{Con}(T) \leftrightarrow \mathrm{Con}^*(T)$,

where $\mathrm{Con}^*(T) :\equiv \neg\exists x \mathrm{Prf}^*_T(x, \overline{\ulcorner\bot\urcorner})$.

Ambiguity

Even if the following hold for all Λ and φ:

- $\mathbf{Q} \vdash \mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is a T-proof of φ and
  $\mathbf{Q} \vdash \neg\mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is not a T-proof of φ;
- $\mathbf{Q} \vdash \mathrm{Prf}^*_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is a T-proof of φ and
  $\mathbf{Q} \vdash \neg\mathrm{Prf}^*_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is not a T-proof of φ;

we do *not* have

- $\mathbf{Q} \vdash \forall x, y(\mathrm{Prf}_T(x, y) \leftrightarrow \mathrm{Prf}^*_T(x, y))$, nor
- $\mathbf{Q} \vdash \mathrm{Con}(T) \leftrightarrow \mathrm{Con}^*(T)$,

where $\mathrm{Con}^*(T) :\equiv \neg\exists x \mathrm{Prf}^*_T(x, \overline{\ulcorner\bot\urcorner})$.

The point here:

$$T \vdash \varphi(\overline{k}) \text{ for all } k \in \omega \;\not\Rightarrow\; T \vdash \forall x \varphi(x).$$

Quiz 3 — Which is correct?

- Gödel 2nd Incompleteness (1931):
  PA *cannot prove a sentence which represents the consistency of* **PA**.

- Kreisel's Remark (1960):
  PA *does prove a sentence which represents the consistency of* **PA**.

2. A Brief Look at the Proofs

Rosser's trick

Given Prf_T such that, for all Λ and φ,

- $\mathbf{Q} \vdash \mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is a T-proof of φ,
- $\mathbf{Q} \vdash \neg\mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is not a T-proof of φ,

Rosser's trick

Given Prf_T such that, for all Λ and φ,

- $\mathbf{Q} \vdash \mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is a T-proof of φ,
- $\mathbf{Q} \vdash \neg\mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is not a T-proof of φ,

we can define Prf^*_T by

$$\mathrm{Prf}^*_T(x, u) :\equiv \mathrm{Prf}(x, u) \wedge$$
$$(\forall z < x)\forall v \neg(\mathrm{neg}(u, v) \wedge \mathrm{Prf}(z, v)).$$

Rosser's trick

Given Prf_T such that, for all Λ and φ,

- $\mathbf{Q} \vdash \mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is a T-proof of φ,
- $\mathbf{Q} \vdash \neg\mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is not a T-proof of φ,

we can define Prf^*_T by

$$\mathrm{Prf}^*_T(x,u) :\equiv \mathrm{Prf}(x,u) \wedge$$
$$(\forall z < x)\forall v \neg(\mathrm{neg}(u,v) \wedge \mathrm{Prf}(z,v)).$$

Then

$\mathbf{Q} \vdash \mathrm{Prf}^*_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is a T-proof of $\varphi \wedge$
there is no T-proof Δ of $\neg\varphi$
with $\ulcorner\Delta\urcorner < \ulcorner\Lambda\urcorner$
$\implies \Lambda$ is a T-proof of φ.

Rosser's trick

Given Prf_T such that, for all Λ and φ,

- **Q** $\vdash \mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is a T-proof of φ,
- **Q** $\vdash \neg\mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is not a T-proof of φ,

we can define Prf^*_T by

$$\mathrm{Prf}^*_T(x,u) :\equiv \mathrm{Prf}(x,u) \wedge \\ (\forall z < x)\forall v \neg(\mathrm{neg}(u,v) \wedge \mathrm{Prf}(z,v)).$$

Then

Q $\vdash \mathrm{Prf}^*_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is a T-proof of φ $\wedge$

there is no T-proof Δ of $\neg\varphi$

with $\ulcorner\Delta\urcorner < \ulcorner\Lambda\urcorner$

$\Longrightarrow \Lambda$ is a T-proof of φ.

If T is consistent, $\Longleftarrow$ also holds.

Rosser's trick

Given Prf_T such that, for all Λ and φ,

- $\mathbf{Q} \vdash \mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is a T-proof of φ,
- $\mathbf{Q} \vdash \neg\mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is not a T-proof of φ,

we can define Prf^*_T by

$$\mathrm{Prf}^*_T(x, u) :\equiv \mathrm{Prf}(x, u) \wedge (\forall z < x)\forall v \neg(\mathrm{neg}(u, v) \wedge \mathrm{Prf}(z, v)).$$

Then

$\mathbf{Q} \vdash \neg\mathrm{Prf}^*_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is not a T-proof of $\varphi \vee$
there is a T-proof Δ of $\neg\varphi$
with $\ulcorner\Delta\urcorner < \ulcorner\Lambda\urcorner$
$\Longleftarrow \Lambda$ is not a T-proof of φ.

Rosser's trick

Given Prf_T such that, for all Λ and φ,

- $\mathbf{Q} \vdash \mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is a T-proof of φ,
- $\mathbf{Q} \vdash \neg\mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is not a T-proof of φ,

we can define Prf^*_T by

$$\mathrm{Prf}^*_T(x,u) :\equiv \mathrm{Prf}(x,u) \wedge (\forall z < x)\forall v \neg(\mathrm{neg}(u,v) \wedge \mathrm{Prf}(z,v)).$$

Then

$\mathbf{Q} \vdash \neg\mathrm{Prf}^*_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\varphi\urcorner}) \iff \Lambda$ is not a T-proof of $\varphi \vee$ there is a T-proof Δ of $\neg\varphi$ with $\ulcorner\Delta\urcorner < \ulcorner\Lambda\urcorner$

$\Longleftarrow \Lambda$ is not a T-proof of φ.

If T is consistent, $\Longrightarrow$ also holds.

Kreisel’s remark (1960)

Since there is a proof Δ of $\neg\bot$, if T is consistent,

$$\mathbf{Q} \vdash (\forall x < \overline{\ulcorner \Delta \urcorner}) \neg \mathrm{Prf}(x, \overline{\ulcorner \bot \urcorner}).$$

Kreisel's remark (1960)

Since there is a proof Δ of $\neg\bot$, if T is consistent,

$$\mathbf{Q} \vdash (\forall x < \overline{\ulcorner \Delta \urcorner})\neg \mathrm{Prf}(x, \overline{\ulcorner \bot \urcorner}).$$

Hence, Q proves

$$\begin{aligned}
\mathrm{Prf}^*_T(x, \overline{\ulcorner \bot \urcorner}) \equiv \mathrm{Prf}_T(x, \overline{\ulcorner \bot \urcorner}) \wedge \\
(\forall z < x)\forall v \neg(\mathrm{neg}(\overline{\ulcorner \bot \urcorner}, v) \wedge \mathrm{Prf}_T(z, v)) \\
\to \forall v \neg(\mathrm{neg}(\overline{\ulcorner \bot \urcorner}, v) \wedge \mathrm{Prf}_T(\overline{\ulcorner \Delta \urcorner}, v)) \\
\leftrightarrow \neg \mathrm{Prf}_T(\overline{\ulcorner \Delta \urcorner}, \overline{\ulcorner \neg\bot \urcorner}) \\
\to \bot.
\end{aligned}$$

Kreisel's remark (1960)

Since there is a proof Δ of $\neg\bot$, if T is consistent,

$$\mathbf{Q} \vdash (\forall x < \overline{\ulcorner \Delta \urcorner})\neg \mathrm{Prf}(x, \overline{\ulcorner \bot \urcorner}).$$

Hence, Q proves

$$\begin{aligned} \mathrm{Prf}^*_T(x, \overline{\ulcorner \bot \urcorner}) \equiv \mathrm{Prf}_T(x, \overline{\ulcorner \bot \urcorner}) \wedge \\ (\forall z < x)\forall v \neg(\mathrm{neg}(\overline{\ulcorner \bot \urcorner}, v) \wedge \mathrm{Prf}_T(z, v)) \\ \rightarrow \forall v \neg(\mathrm{neg}(\overline{\ulcorner \bot \urcorner}, v) \wedge \mathrm{Prf}_T(\overline{\ulcorner \Delta \urcorner}, v)) \\ \leftrightarrow \neg \mathrm{Prf}_T(\overline{\ulcorner \Delta \urcorner}, \overline{\ulcorner \neg\bot \urcorner}) \\ \rightarrow \bot. \end{aligned}$$

For any consistent recursively axiomatizable T,

$$\mathbf{Q} \vdash \mathrm{Con}^*(T).$$

The Statement (6)

If a first order theory T satisfies the following:

- T is consistent;
- T is recursively axiomatizable;
- T essentially contains Robinson Arithmetic Q,

then the following hold:

1st incompleteness: T is not complete;

2nd incompleteness: T cannot prove a sentence which represents the consistency of T.

Gödel's result

If a first order theory T satisfies the following:

- T is ω-consistent;
- T is recursively axiomatizable;
- T essentially contains Robinson Arithmetic **Q**,

then the following hold:

1st incompleteness: T is not complete;

2nd incompleteness: T cannot prove a sentence which represents the consistency of T.

Gödel's result

If a first order theory T satisfies the following:

- T is ω-consistent;
- T is recursively axiomatizable;
- T essentially contains Robinson Arithmetic **Q**,

then the following hold:

1st incompleteness: T is not complete;

2nd incompleteness: T cannot prove a sentence which represents the consistency of T.

T is called ω-consistent iff there is no $\varphi(x) \in L_T$ s.t.

- $T \vdash \neg\varphi(\overline{k})$ for all $k \in \omega$;
- $T \vdash \exists x \varphi(x)$.

Gödel's Self-reference Lemma

Lemma For any $\varphi(x) \in L_{\mathbf{Q}}$, there is a $L_{\mathbf{Q}}$ sentence θ s.t.

$$\mathbf{Q} \vdash \theta \leftrightarrow \varphi(\overline{\ulcorner \theta \urcorner}).$$

Gödel's Self-reference Lemma

Lemma For any $\varphi(x) \in L_{\mathbf{Q}}$, there is a $L_{\mathbf{Q}}$ sentence θ s.t.

$$\mathbf{Q} \vdash \theta \leftrightarrow \varphi(\overline{\ulcorner \theta \urcorner}).$$

(proof) Take $\mathrm{Subst}(u, y, v)$ such that

- $\mathbf{Q} \vdash \mathrm{Subst}(\overline{\ulcorner \tau(x) \urcorner}, \overline{\ulcorner t \urcorner}, \overline{k}) \iff k = \ulcorner \tau(t) \urcorner$;
- $\mathbf{Q} \vdash \neg\mathrm{Subst}(\overline{\ulcorner \tau(x) \urcorner}, \overline{\ulcorner t \urcorner}, \overline{k}) \iff k \neq \ulcorner \tau(t) \urcorner$;

Gödel's Self-reference Lemma

Lemma For any $\varphi(x) \in L_{\mathbf{Q}}$, there is a $L_{\mathbf{Q}}$ sentence θ s.t.

$$\mathbf{Q} \vdash \theta \leftrightarrow \varphi(\overline{\ulcorner \theta \urcorner}).$$

(proof) Take $\mathrm{Subst}(u, y, v)$ such that

- $\mathbf{Q} \vdash \mathrm{Subst}(\overline{\ulcorner \tau(x) \urcorner}, \overline{\ulcorner t \urcorner}, \overline{k}) \iff k = \ulcorner \tau(t) \urcorner$;
- $\mathbf{Q} \vdash \neg\mathrm{Subst}(\overline{\ulcorner \tau(x) \urcorner}, \overline{\ulcorner t \urcorner}, \overline{k}) \iff k \neq \ulcorner \tau(t) \urcorner$;

Let $\rho(x) :\equiv \exists u(\mathrm{Subst}(x, \overline{\ulcorner x \urcorner}, u) \wedge \varphi(u))$.

Gödel's Self-reference Lemma

Lemma For any $\varphi(x) \in L_{\mathbf{Q}}$, there is a $L_{\mathbf{Q}}$ sentence θ s.t.

$$\mathbf{Q} \vdash \theta \leftrightarrow \varphi(\overline{\ulcorner\theta\urcorner}).$$

(proof) Take $\mathrm{Subst}(u, y, v)$ such that

- $\mathbf{Q} \vdash \mathrm{Subst}(\overline{\ulcorner\tau(x)\urcorner}, \overline{\ulcorner t\urcorner}, \overline{k}) \iff k = \ulcorner\tau(t)\urcorner$;
- $\mathbf{Q} \vdash \neg\mathrm{Subst}(\overline{\ulcorner\tau(x)\urcorner}, \overline{\ulcorner t\urcorner}, \overline{k}) \iff k \neq \ulcorner\tau(t)\urcorner$;

Let $\rho(x) :\equiv \exists u(\mathrm{Subst}(x, \overline{\ulcorner x\urcorner}, u) \wedge \varphi(u))$.
For any $\tau(x)$,

$$\mathbf{Q} \vdash \rho(\overline{\ulcorner\tau(x)\urcorner}) \leftrightarrow \varphi(\overline{\ulcorner\tau(\overline{\ulcorner\tau(x)\urcorner})\urcorner}).$$

Gödel's Self-reference Lemma

Lemma For any $\varphi(x) \in L_{\mathbf{Q}}$, there is a $L_{\mathbf{Q}}$ sentence θ s.t.

$$\mathbf{Q} \vdash \theta \leftrightarrow \varphi(\overline{\ulcorner\theta\urcorner}).$$

(proof) Take $\mathrm{Subst}(u, y, v)$ such that

- $\mathbf{Q} \vdash \mathrm{Subst}(\overline{\ulcorner\tau(x)\urcorner}, \overline{\ulcorner t\urcorner}, \overline{k}) \iff k = \ulcorner\tau(t)\urcorner$;
- $\mathbf{Q} \vdash \neg\mathrm{Subst}(\overline{\ulcorner\tau(x)\urcorner}, \overline{\ulcorner t\urcorner}, \overline{k}) \iff k \neq \ulcorner\tau(t)\urcorner$;

Let $\rho(x) :\equiv \exists u(\mathrm{Subst}(x, \overline{\ulcorner x\urcorner}, u) \wedge \varphi(u))$.
For any $\tau(x)$,

$$\mathbf{Q} \vdash \rho(\overline{\ulcorner\tau(x)\urcorner}) \leftrightarrow \varphi(\overline{\ulcorner\tau(\overline{\ulcorner\tau(x)\urcorner})\urcorner}).$$

Letting $\tau(x) \equiv \rho(x)$ and $\theta \equiv \rho(\overline{\ulcorner\rho(x)\urcorner})$, we have

$$\mathbf{Q} \vdash \rho(\overline{\ulcorner\rho(x)\urcorner}) \leftrightarrow \varphi(\overline{\ulcorner\rho(\overline{\ulcorner\rho(x)\urcorner})\urcorner}).$$

Gödel's 1st incompleteness

Theorem If T is ω-consistent, ...(omitted)...,
then there is $\sigma \in L_{\mathbf{Q}}$ s.t. $T \nvdash \sigma$ and $T \nvdash \neg\sigma$.

Gödel's 1st incompleteness

Theorem If T is ω-consistent, ...(omitted)..., then there is $\sigma \in L_{\mathbf{Q}}$ s.t. $T \nvdash \sigma$ and $T \nvdash \neg\sigma$.

(Proof) By self-reference lemma, we have σ s.t.

$$\mathbf{Q} \vdash \sigma \leftrightarrow \neg\exists x \mathrm{Prf}_T(x, \overline{\ulcorner\sigma\urcorner}).$$

Gödel's 1st incompleteness

Theorem If T is ω-consistent, ...(omitted)...,
then there is $\sigma \in L_{\mathbf{Q}}$ s.t. $T \nvdash \sigma$ and $T \nvdash \neg\sigma$.

(Proof) By self-reference lemma, we have σ s.t.

$$\mathbf{Q} \vdash \sigma \leftrightarrow \neg\exists x \mathrm{Prf}_T(x, \overline{\ulcorner\sigma\urcorner}).$$

Suppose $T \vdash \sigma$. There is a T-proof Λ of σ.
Then $\mathbf{Q} \vdash \mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\sigma\urcorner})$, and $\mathbf{Q} \vdash \exists x \mathrm{Prf}_T(x, \overline{\ulcorner\sigma\urcorner})$.
Thus $\mathbf{Q} \vdash \neg\sigma$, contradicting the consistency of T.

Gödel's 1st incompleteness

Theorem If T is ω-consistent, ...(omitted)...,
then there is $\sigma \in L_{\mathbf{Q}}$ s.t. $T \nvdash \sigma$ and $T \nvdash \neg\sigma$.

(Proof) By self-reference lemma, we have σ s.t.

$$\mathbf{Q} \vdash \sigma \leftrightarrow \neg\exists x \mathrm{Prf}_T(x, \overline{\ulcorner\sigma\urcorner}).$$

Suppose $T \vdash \sigma$. There is a T-proof Λ of σ.
Then $\mathbf{Q} \vdash \mathrm{Prf}_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\sigma\urcorner})$, and $\mathbf{Q} \vdash \exists x \mathrm{Prf}_T(x, \overline{\ulcorner\sigma\urcorner})$.
Thus $\mathbf{Q} \vdash \neg\sigma$, contradicting the consistency of T.

Suppose $T \vdash \neg\sigma$. Then $T \nvdash \sigma$ by consistency.
Thus $\mathbf{Q} \vdash \neg\mathrm{Prf}_T(\overline{k}, \overline{\ulcorner\sigma\urcorner})$ for all $k \in \omega$.
However, $T \vdash \exists x \mathrm{Prf}_T(x, \overline{\ulcorner\sigma\urcorner})$,
and so T is ω-inconsistent.

Rosser's enhancement

Theorem If T is consistent, ...(omitted)...,
then there is $\sigma \in L_{\mathbf{Q}}$ s.t. $T \nvdash \sigma$ and $T \nvdash \neg\sigma$.

(Proof) By self-reference lemma, we have σ s.t.

$$\mathbf{Q} \vdash \sigma \leftrightarrow \neg\exists x \mathrm{Prf}_T^*(x, \overline{\ulcorner\sigma\urcorner}).$$

Suppose $T \vdash \sigma$. There is a T-proof Λ of σ.
Then $\mathbf{Q} \vdash \mathrm{Prf}_T^*(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\sigma\urcorner})$, and $\mathbf{Q} \vdash \exists x \mathrm{Prf}_T^*(x, \overline{\ulcorner\sigma\urcorner})$.
Thus $\mathbf{Q} \vdash \neg\sigma$, contradicting the consistency of T.

Rosser's enhancement

Theorem If T is consistent, ...(omitted)...,
then there is $\sigma \in L_{\mathbf{Q}}$ s.t. $T \nvdash \sigma$ and $T \nvdash \neg\sigma$.

(Proof) By self-reference lemma, we have σ s.t.

$$\mathbf{Q} \vdash \sigma \leftrightarrow \neg\exists x \mathrm{Prf}^*_T(x, \overline{\ulcorner\sigma\urcorner}).$$

Suppose $T \vdash \sigma$. There is a T-proof Λ of σ.
Then $\mathbf{Q} \vdash \mathrm{Prf}^*_T(\overline{\ulcorner\Lambda\urcorner}, \overline{\ulcorner\sigma\urcorner})$, and $\mathbf{Q} \vdash \exists x \mathrm{Prf}^*_T(x, \overline{\ulcorner\sigma\urcorner})$.
Thus $\mathbf{Q} \vdash \neg\sigma$, contradicting the consistency of T.

Suppose $T \vdash \neg\sigma$. So $\mathbf{Q} \vdash \mathrm{Prf}^*_T(\overline{\ulcorner\Delta\urcorner}, \overline{\ulcorner\neg\sigma\urcorner})$ for some Δ.
Since T is consistent, $\mathbf{Q} \vdash (\forall x < \overline{\ulcorner\Delta\urcorner})\neg\mathrm{Prf}_T(x, \overline{\ulcorner\sigma\urcorner})$.
But $T \vdash \exists x \mathrm{Prf}^*_T(x, \overline{\ulcorner\sigma\urcorner})$, i.e.,
$T \vdash \exists x(\mathrm{Prf}_T(x, \overline{\ulcorner\sigma\urcorner}) \wedge (\forall y < x)\neg\mathrm{Prf}_T(y, \overline{\ulcorner\neg\sigma\urcorner}))$.

A dilemma

- To obtain the incompleteness without ω-consistency but only consistency, the key is Rosser's modification Prf^*_T for representing the notion "... is a proof of ...";

A dilemma

- To obtain the incompleteness without ω-consistency but only consistency, the key is Rosser's modification Prf^*_T for representing the notion "... is a proof of ...";

- but the corresponding consistency statement $\mathrm{Con}^*(T)$ is provable even in **Q** and hence in T.

Loeb's derivability conditions

A "canonicality" on $\mathrm{Pv}_T(u) \equiv \exists x \mathrm{Prf}_T(x, u)$:

(1) If $T \vdash \varphi$ then $\mathbf{Q} \vdash \mathrm{Pv}_T(\overline{\ulcorner \varphi \urcorner})$;

(2) $\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1 \vdash \mathrm{Pv}_T(\overline{\ulcorner \varphi \to \psi \urcorner}) \to (\mathrm{Pv}_T(\overline{\ulcorner \varphi \urcorner}) \to \mathrm{Pv}_T(\overline{\ulcorner \psi \urcorner}))$

(3) $\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1 \vdash \mathrm{Pv}_T(\overline{\ulcorner \varphi \urcorner}) \to \mathrm{Pv}_T(\overline{\ulcorner \mathrm{Pv}_T(\overline{\ulcorner \varphi \urcorner}) \urcorner})$.

Loeb's derivability conditions

A "canonicality" on $\mathrm{Pv}_T(u) \equiv \exists x \mathrm{Prf}_T(x, u)$:

(1) If $T \vdash \varphi$ then $\mathbf{Q} \vdash \mathrm{Pv}_T(\overline{\ulcorner \varphi \urcorner})$;

(2) $\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1 \vdash \mathrm{Pv}_T(\overline{\ulcorner \varphi \to \psi \urcorner}) \to (\mathrm{Pv}_T(\overline{\ulcorner \varphi \urcorner}) \to \mathrm{Pv}_T(\overline{\ulcorner \psi \urcorner}))$

(3) $\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1 \vdash \mathrm{Pv}_T(\overline{\ulcorner \varphi \urcorner}) \to \mathrm{Pv}_T(\overline{\ulcorner \mathrm{Pv}_T(\overline{\ulcorner \varphi \urcorner}) \urcorner})$.

These conditions imply $T \nvdash \neg \mathrm{Pv}_T(\overline{\ulcorner \bot \urcorner})$.

Loeb's derivability conditions

A "canonicality" on $\mathrm{Pv}_T(u) \equiv \exists x \mathrm{Prf}_T(x, u)$:

(1) If $T \vdash \varphi$ then $\mathbf{Q} \vdash \mathrm{Pv}_T(\overline{\ulcorner \varphi \urcorner})$;

(2) $\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1 \vdash \mathrm{Pv}_T(\overline{\ulcorner \varphi \to \psi \urcorner}) \to (\mathrm{Pv}_T(\overline{\ulcorner \varphi \urcorner}) \to \mathrm{Pv}_T(\overline{\ulcorner \psi \urcorner}))$

(3) $\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1 \vdash \mathrm{Pv}_T(\overline{\ulcorner \varphi \urcorner}) \to \mathrm{Pv}_T(\overline{\ulcorner \mathrm{Pv}_T(\overline{\ulcorner \varphi \urcorner}) \urcorner})$.

These conditions imply $T \nvdash \neg \mathrm{Pv}_T(\overline{\ulcorner \bot \urcorner})$.

(Proof) Self-reference Lemma yields σ s.t.

$$\mathbf{Q} \vdash \sigma \leftrightarrow (\mathrm{Pv}_T(\overline{\ulcorner \sigma \urcorner}) \to \bot).$$

Loeb's derivability conditions

A "canonicality" on $\mathrm{Pv}_T(u) \equiv \exists x \mathrm{Prf}_T(x,u)$:

(1) If $T \vdash \varphi$ then $\mathbf{Q} \vdash \mathrm{Pv}_T(\overline{\ulcorner\varphi\urcorner})$;

(2) $\mathbf{I\Sigma}_0+\mathbf{\Omega}_1 \vdash \mathrm{Pv}_T(\overline{\ulcorner\varphi\to\psi\urcorner}) \to (\mathrm{Pv}_T(\overline{\ulcorner\varphi\urcorner}) \to \mathrm{Pv}_T(\overline{\ulcorner\psi\urcorner}))$

(3) $\mathbf{I\Sigma}_0+\mathbf{\Omega}_1 \vdash \mathrm{Pv}_T(\overline{\ulcorner\varphi\urcorner}) \to \mathrm{Pv}_T(\overline{\ulcorner\mathrm{Pv}_T(\overline{\ulcorner\varphi\urcorner})\urcorner})$.

These conditions imply $T \nvdash \neg\mathrm{Pv}_T(\overline{\ulcorner\bot\urcorner})$.

(Proof) Self-reference Lemma yields σ s.t.

$$\mathbf{Q} \vdash \sigma \leftrightarrow (\mathrm{Pv}_T(\overline{\ulcorner\sigma\urcorner}) \to \bot).$$

Then the conditions (1) and (2) yield

$$\mathbf{I\Sigma}_0+\mathbf{\Omega}_1 \vdash \mathrm{Pv}_T(\overline{\ulcorner\sigma\urcorner}) \to (\mathrm{Pv}_T(\overline{\ulcorner\mathrm{Pv}_T(\overline{\ulcorner\sigma\urcorner})\urcorner}) \to \mathrm{Pv}_T(\overline{\ulcorner\bot\urcorner})).$$

Loeb's derivability conditions

A "canonicality" on $\mathrm{Pv}_T(u) \equiv \exists x \mathrm{Prf}_T(x, u)$:

(1) If $T \vdash \varphi$ then $\mathbf{Q} \vdash \mathrm{Pv}_T(\overline{\ulcorner \varphi \urcorner})$;

(2) $\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1 \vdash \mathrm{Pv}_T(\overline{\ulcorner \varphi \to \psi \urcorner}) \to (\mathrm{Pv}_T(\overline{\ulcorner \varphi \urcorner}) \to \mathrm{Pv}_T(\overline{\ulcorner \psi \urcorner}))$

(3) $\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1 \vdash \mathrm{Pv}_T(\overline{\ulcorner \varphi \urcorner}) \to \mathrm{Pv}_T(\overline{\ulcorner \mathrm{Pv}_T(\overline{\ulcorner \varphi \urcorner}) \urcorner})$.

These conditions imply $T \nvdash \neg \mathrm{Pv}_T(\overline{\ulcorner \bot \urcorner})$.

(Proof) Self-reference Lemma yields σ s.t.

$$\mathbf{Q} \vdash \sigma \leftrightarrow (\mathrm{Pv}_T(\overline{\ulcorner \sigma \urcorner}) \to \bot).$$

Then the conditions (1) and (2) yield

$\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1 \vdash \mathrm{Pv}_T(\overline{\ulcorner \sigma \urcorner}) \to (\mathrm{Pv}_T(\overline{\ulcorner \mathrm{Pv}_T(\overline{\ulcorner \sigma \urcorner}) \urcorner}) \to \mathrm{Pv}_T(\overline{\ulcorner \bot \urcorner}))$.

(3) yields $\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1 \vdash \mathrm{Pv}_T(\overline{\ulcorner \sigma \urcorner}) \to \mathrm{Pv}_T(\overline{\ulcorner \bot \urcorner})$, and so

$\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1 \vdash \neg \mathrm{Pv}_T(\overline{\ulcorner \bot \urcorner}) \to \sigma$. Since $T \nvdash \sigma$, done!

A controversy

In the case of $T = \mathbf{Q}$:

- everything must be through Nelson's interpretation $\mathbf{n}$ of $\mathbf{I\Sigma_0{+}\Omega_1}$ in $\mathbf{Q}$;

A controversy

In the case of $T = \mathbf{Q}$:

- everything must be through Nelson's interpretation $\mathbf{n}$ of $\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1$ in $\mathbf{Q}$;
- $\mathrm{Pv}_T(u)$ is $(\exists x \mathrm{Prf}(x, u))^{\mathbf{n}} \equiv \exists x (W(x) \wedge \mathrm{Prf}(x, u)^{\mathbf{n}})$; and $\mathrm{Con}(T)$ is $\neg \exists x (W(x) \wedge \mathrm{Prf}(x, \overline{\ulcorner \bot \urcorner})^{\mathbf{n}})$;

A controversy

In the case of $T = \mathbf{Q}$:

- everything must be through Nelson's interpretation $\mathbf{n}$ of $\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1$ in $\mathbf{Q}$;
- $\mathrm{Pv}_T(u)$ is $(\exists x \mathrm{Prf}(x, u))^{\mathbf{n}} \equiv \exists x (W(x) \wedge \mathrm{Prf}(x, u)^{\mathbf{n}})$; and $\mathrm{Con}(T)$ is $\neg \exists x (W(x) \wedge \mathrm{Prf}(x, \overline{\ulcorner \bot \urcorner})^{\mathbf{n}})$;
- Does this $\mathrm{Con}(T)$ really represent "consistency"?

A controversy

In the case of $T = \mathbf{Q}$:

- everything must be through Nelson's interpretation $\mathbf{n}$ of $\mathbf{I\Sigma_0{+}\Omega_1}$ in $\mathbf{Q}$;
- $\mathrm{Pv}_T(u)$ is $(\exists x \mathrm{Prf}(x,u))^{\mathbf{n}} \equiv \exists x (W(x) \wedge \mathrm{Prf}(x,u)^{\mathbf{n}})$; and $\mathrm{Con}(T)$ is $\neg\exists x (W(x) \wedge \mathrm{Prf}(x, \overline{\ulcorner \bot \urcorner})^{\mathbf{n}})$;
- Does this $\mathrm{Con}(T)$ really represent "consistency"?

On the other hand, in the case of $T = \mathbf{ZFC}$,

- everything must be through von Neumann's interpretation $\mathbf{v}$ of $\mathbf{I\Sigma_0{+}\Omega_1}$ in $\mathbf{ZFC}$;

A controversy

In the case of $T = \mathbf{Q}$:

- everything must be through Nelson's interpretation $\mathbf{n}$ of $\mathbf{I\Sigma}_0+\mathbf{\Omega}_1$ in $\mathbf{Q}$;
- $\mathrm{Pv}_T(u)$ is $(\exists x \mathrm{Prf}(x,u))^{\mathbf{n}} \equiv \exists x(W(x) \land \mathrm{Prf}(x,u)^{\mathbf{n}})$; and $\mathrm{Con}(T)$ is $\neg\exists x(W(x) \land \mathrm{Prf}(x, \overline{\ulcorner\bot\urcorner})^{\mathbf{n}})$;
- Does this $\mathrm{Con}(T)$ really represent "consistency"?

On the other hand, in the case of $T = \mathbf{ZFC}$,

- everything must be through von Neumann's interpretation $\mathbf{v}$ of $\mathbf{I\Sigma}_0+\mathbf{\Omega}_1$ in $\mathbf{ZFC}$;
- $\mathrm{Pv}_T(u)$ is $(\exists x \mathrm{Prf}(x,u))^{\mathbf{v}} \equiv (\exists x \in \omega)\mathrm{Prf}(x,u)^{\mathbf{v}}$; and $\mathrm{Con}(T)$ is $\neg(\exists x \in \omega)\mathrm{Prf}(x, \overline{\ulcorner\bot\urcorner})^{\mathbf{v}}$.

A controversy

In the case of $T = \mathbf{Q}$:

- everything must be through Nelson's interpretation $\mathbf{n}$ of $\mathbf{I\Sigma_0{+}\Omega_1}$ in $\mathbf{Q}$;
- $\mathrm{Pv}_T(u)$ is $(\exists x \mathrm{Prf}(x,u))^{\mathbf{n}} \equiv \exists x (W(x) \wedge \mathrm{Prf}(x,u)^{\mathbf{n}})$; and $\mathrm{Con}(T)$ is $\neg \exists x (W(x) \wedge \mathrm{Prf}(x, \overline{\ulcorner \bot \urcorner})^{\mathbf{n}})$;
- Does this $\mathrm{Con}(T)$ really represent "consistency"?

On the other hand, in the case of $T = \mathbf{ZFC}$,

- everything must be through von Neumann's interpretation $\mathbf{v}$ of $\mathbf{I\Sigma_0{+}\Omega_1}$ in $\mathbf{ZFC}$;
- $\mathrm{Pv}_T(u)$ is $(\exists x \mathrm{Prf}(x,u))^{\mathbf{v}} \equiv (\exists x \in \omega) \mathrm{Prf}(x,u)^{\mathbf{v}}$; and $\mathrm{Con}(T)$ is $\neg (\exists x \in \omega) \mathrm{Prf}(x, \overline{\ulcorner \bot \urcorner})^{\mathbf{v}}$.

What's the difference between them?

3. Connection to the present-day researches

Comparison of thoeries

Given $S \subseteq T$, under which condition, a theory T could be said essentially stronger than another S?

Comparison of thoeries

Given $S \subseteq T$, under which condition, a theory T could be said essentially stronger than another S?

(A) there is a sentence φ s.t. $S \nvdash \varphi$ and $T \vdash \varphi$?

- by changing ways of formalizing concepts, S might be able to simulate T;
- e.g., $\mathbf{ZFC}-\mathbf{FA}$ can simulate $\mathbf{ZFC}$, and $\mathbf{ZFC}-\mathbf{Ext}$ can simulate $\mathbf{ZFC}$.

Comparison of thoeries

Given $S \subseteq T$, under which condition, a theory T could be said essentially stronger than another S?

(A) there is a sentence φ s.t. $S \nvdash \varphi$ and $T \vdash \varphi$?

- by changing ways of formalizing concepts, S might be able to simulate T;
- e.g., $\mathbf{ZFC}-\mathbf{FA}$ can simulate $\mathbf{ZFC}$, and $\mathbf{ZFC}-\mathbf{Ext}$ can simulate $\mathbf{ZFC}$.

(B) there is no interpretation of T in S?

- prevents the possibility that S simulates T;

Comparison of thoeries

Given $S \subseteq T$, under which condition, a theory T could be said essentially stronger than another S?

(A) there is a sentence φ s.t. $S \nvdash \varphi$ and $T \vdash \varphi$?

- by changing ways of formalizing concepts, S might be able to simulate T;
- e.g., $\mathbf{ZFC}-\mathbf{FA}$ can simulate $\mathbf{ZFC}$, and $\mathbf{ZFC}-\mathbf{Ext}$ can simulate $\mathbf{ZFC}$.

(B) there is no interpretation of T in S?

- prevents the possibility that S simulates T;

While there is another way to obtain (A), e.g., constructing a model M s.t. $M \models S$ but $M \not\models T$,

Comparison of thoeries

Given $S \subseteq T$, under which condition, a theory T could be said essentially stronger than another S?

(A) there is a sentence φ s.t. $S \nvdash \varphi$ and $T \vdash \varphi$?

- by changing ways of formalizing concepts, S might be able to simulate T;
- e.g., **ZFC**$-$**FA** can simulate **ZFC**, and **ZFC**$-$**Ext** can simulate **ZFC**.

(B) there is no interpretation of T in S?

- prevents the possibility that S simulates T;

While there is another way to obtain (A), e.g.,

constructing a model M s.t. $M \models S$ but $M \not\models T$,

practically the only way to obtain (B) is

showing $T \vdash \mathrm{Con}(S)$.

Gödel hierarchy

For theories T and S which are consistent, recursively axiomatizable, essentially containing $\mathbf{Q}$,

- $S < T$ iff $T \vdash \mathrm{Con}(S)$;
- $S \equiv T$ iff $\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1 \vdash \mathrm{Con}(S) \leftrightarrow \mathrm{Con}(T)$.

Gödel hierarchy

For theories T and S which are consistent, recursively axiomatizable, essentially containing $\mathbf{Q}$,

- $S < T$ iff $T \vdash \mathrm{Con}(S)$;
- $S \equiv T$ iff $\mathbf{I\Sigma}_0+\mathbf{\Omega}_1 \vdash \mathrm{Con}(S) \leftrightarrow \mathrm{Con}(T)$.

Large parts of proof theory and set theory are investigations of this hierarchy:

Gödel hierarchy

For theories T and S which are consistent, recursively axiomatizable, essentially containing $\mathbf{Q}$,

- $S < T$ iff $T \vdash \mathrm{Con}(S)$;
- $S \equiv T$ iff $\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1 \vdash \mathrm{Con}(S) \leftrightarrow \mathrm{Con}(T)$.

Large parts of proof theory and set theory are investigations of this hierarchy:

- measure for $<$:
  proof theoretic ordinal; large cardinal.

Gödel hierarchy

For theories T and S which are consistent, recursively axiomatizable, essentially containing $\mathbf{Q}$,

- $S < T$ iff $T \vdash \mathrm{Con}(S)$;
- $S \equiv T$ iff $\mathbf{I\Sigma}_0 + \mathbf{\Omega}_1 \vdash \mathrm{Con}(S) \leftrightarrow \mathrm{Con}(T)$.

Large parts of proof theory and set theory are investigations of this hierarchy:

- measure for $<$:
  proof theoretic ordinal; large cardinal.
- methods establishing $\equiv$:
  cut elimination; forcing; inner model, etc.

Picture of the hierarchy

$$
\begin{array}{c}
\mathbf{Z_2} \equiv \mathbf{ZFC}{-}\mathbf{Pow} \\
\vdots \\
\vee \\
\Pi^1_2\text{-}\mathbf{CA}_0 \\
\vee \\
\Sigma^1_2\text{-}\mathbf{AC} \equiv \mathbf{KPi} \equiv \mathbf{T}_0 \\
\vee \\
\mathbf{ID}_{<\omega} \equiv \Pi^1_1\text{-}\mathbf{CA}_0 \\
\vee \\
\mathbf{ID}_1 \equiv \mathbf{BI} \equiv \mathbf{KP} \equiv \mathbf{CZF} \equiv \mathbf{MLT} \\
\vee \\
\widehat{\mathbf{ID}}_{<\omega} \equiv \mathbf{IR} \equiv \mathbf{ATR}_0 \\
\vee \\
\mathbf{PA} \equiv \mathbf{ACA}_0 \equiv \Sigma^1_1\text{-}\mathbf{AC}_0 \equiv \mathbf{HA} \\
\vdots \\
\vee \\
\mathbf{I\Sigma}_2 \\
\vee \\
\mathbf{PRA} \equiv \mathbf{I\Sigma}_1 \equiv \mathbf{RCA}_0 \equiv \mathbf{WKL}_0 \\
\vee \\
\mathbf{Q} \equiv \mathbf{I\Sigma}_0 + \mathbf{\Omega}_1
\end{array}
$$

Picture of the hierarchy

$$\mathbf{Z_2} \equiv \mathbf{ZFC}{-}\mathbf{Pow}$$
$$\vdots$$
$$\vee$$
$$\Pi^1_2\text{-}\mathbf{CA}_0$$
$$\vee$$
$$\Sigma^1_2\text{-}\mathbf{AC} \equiv \mathbf{KPi} \equiv \mathbf{T}_0$$
$$\vee$$
$$\mathbf{ID}_{<\omega} \equiv \Pi^1_1\text{-}\mathbf{CA}_0$$
$$\vee$$
$$\mathbf{ID}_1 \equiv \mathbf{BI} \equiv \mathbf{KP} \equiv \mathbf{CZF} \equiv \mathbf{MLT}$$
$$\vee$$
$$\widehat{\mathbf{ID}}_{<\omega} \equiv \mathbf{IR} \equiv \mathbf{ATR}_0$$
$$\vee$$
$$\mathbf{PA} \equiv \mathbf{ACA}_0 \equiv \Sigma^1_1\text{-}\mathbf{AC}_0 \equiv \mathbf{HA}$$
$$\vdots$$
$$\vee$$
$$\mathbf{I\Sigma}_2$$
$$\vee$$
$$\mathbf{PRA} \equiv \mathbf{I\Sigma}_1 \equiv \mathbf{RCA}_0 \equiv \mathbf{WKL}_0$$
$$\vee$$
$$\mathbf{Q} \equiv \mathbf{I\Sigma}_0{+}\mathbf{\Omega}_1$$

$$\mathbf{ZFC}{+}\mathbf{Inac}$$
$$\vdots$$
$$\vee$$
$$\mathbf{ZFC}^3$$
$$\vee$$
$$\mathbf{MK}(:= \mathbf{ZFC}^2)$$
$$\vdots$$
$$\vee$$
$$\mathbf{NBG}{+}\Pi^1_1\text{-}\mathbf{CA}$$
$$\vee$$
$$\mathbf{NBG}{+}\mathbf{ETR}$$
$$\vee$$
$$\mathbf{ZF} \equiv \mathbf{ZFC} \equiv \mathbf{NBG}$$
$$\vee$$
$$\mathbf{Z}$$
$$\vee$$
$$\mathbf{Z}_{<\omega} \equiv \mathbf{ZBQC}$$
$$\vdots$$
$$\vee$$
$$\mathbf{Z}_3$$
$$\vee$$
$$\mathbf{Z_2} \equiv \mathbf{ZFC}{-}\mathbf{Pow}$$

Picture of the hierarchy

$$\mathbf{Z_2} \equiv \mathbf{ZFC{-}Pow}$$
$$\vdots$$
$$\vee$$
$$\Pi^1_2\textbf{-CA}_0$$
$$\vee$$
$$\Sigma^1_2\textbf{-AC} \equiv \mathbf{KPi} \equiv \mathbf{T}_0$$
$$\vee$$
$$\mathbf{ID}_{<\omega} \equiv \Pi^1_1\textbf{-CA}_0$$
$$\vee$$
$$\mathbf{ID}_1 \equiv \mathbf{BI} \equiv \mathbf{KP} \equiv \mathbf{CZF} \equiv \mathbf{MLT}$$
$$\vee$$
$$\widehat{\mathbf{ID}}_{<\omega} \equiv \mathbf{IR} \equiv \mathbf{ATR}_0$$
$$\vee$$
$$\mathbf{PA} \equiv \mathbf{ACA}_0 \equiv \Sigma^1_1\textbf{-AC}_0 \equiv \mathbf{HA}$$
$$\vdots$$
$$\vee$$
$$\mathbf{I\Sigma}_2$$
$$\vee$$
$$\mathbf{PRA} \equiv \mathbf{I\Sigma}_1 \equiv \mathbf{RCA}_0 \equiv \mathbf{WKL}_0$$
$$\vee$$
$$\mathbf{Q} \equiv \mathbf{I\Sigma}_0 + \mathbf{\Omega}_1$$

$$\mathbf{ZFC{+}Inac}$$
$$\vdots$$
$$\vee$$
$$\mathbf{ZFC}^3$$
$$\vee$$
$$\mathbf{MK}(:= \mathbf{ZFC}^2)$$
$$\vdots$$
$$\vee$$
$$\mathbf{NBG} + \Pi^1_1\textbf{-CA}$$
$$\vee$$
$$\mathbf{NBG{+}ETR}$$
$$\vee$$
$$\mathbf{ZF} \equiv \mathbf{ZFC} \equiv \mathbf{NBG}$$
$$\vee$$
$$\mathbf{Z}$$
$$\vee$$
$$\mathbf{Z}_{<\omega} \equiv \mathbf{ZBQC}$$
$$\vdots$$
$$\vee$$
$$\mathbf{Z}_3$$
$$\vee$$
$$\mathbf{Z_2} \equiv \mathbf{ZFC{-}Pow}$$

$$\mathbf{ZFC}+\text{“}0=1\text{”}$$
$$\vdots$$
$$\vee$$
$$\mathbf{ZFC{+}Vop}$$
$$\vee$$
$$\mathbf{ZFC{+}SCpt}$$
$$\vee$$
$$\mathbf{ZFC{+}Wood}$$
$$\vee$$
$$\mathbf{ZFC{+}Meas}$$
$$\vee$$
$$\mathbf{ZFC}+0^\sharp$$
$$\vee$$
$$\mathbf{ZFC{+}WCpt}$$
$$\vdots$$
$$\vee$$
$$\mathbf{ZFC}+2\textbf{-Mahlo}$$
$$\vee$$
$$\mathbf{ZFC{+}Mahlo}$$
$$\vdots$$
$$\vee$$
$$\mathbf{ZFC}+\omega\textbf{-Inac}$$
$$\vdots$$
$$\vee$$
$$\mathbf{ZFC}+2\textbf{-Inac}$$
$$\vee$$
$$\mathbf{ZFC{+}Inac}$$

Picture of the hierarchy

$$\begin{array}{c}
\mathbf{Z_2} \equiv \mathbf{ZFC{-}Pow} \\
\vdots \\
\vee \\
\Pi^1_2\text{-}\mathbf{CA}_0 \\
\vee \\
\Sigma^1_2\text{-}\mathbf{AC} \equiv \mathbf{KPi} \equiv \mathbf{T}_0 \\
\vee \\
\mathbf{ID}_{<\omega} \equiv \Pi^1_1\text{-}\mathbf{CA}_0 \\
\vee \\
\mathbf{ID}_1 \equiv \mathbf{BI} \equiv \mathbf{KP} \equiv \mathbf{CZF} \equiv \mathbf{MLT} \\
\vee \\
\widehat{\mathbf{ID}}_{<\omega} \equiv \mathbf{IR} \equiv \mathbf{ATR}_0 \\
\vee \\
\mathbf{PA} \equiv \mathbf{ACA}_0 \equiv \Sigma^1_1\text{-}\mathbf{AC}_0 \equiv \mathbf{HA} \\
\vdots \\
\vee \\
\mathbf{I}\Sigma_2 \\
\vee \\
\mathbf{PRA} \equiv \mathbf{I}\Sigma_1 \equiv \mathbf{RCA}_0 \equiv \mathbf{WKL}_0 \\
\vee \\
\mathbf{Q} \equiv \mathbf{I}\Sigma_0 + \Omega_1
\end{array}$$

$$\begin{array}{c}
\mathbf{ZFC{+}Inac} \\
\vdots \\
\vee \\
\mathbf{ZFC}^3 \\
\vee \\
\hline
\mathbf{MK}(:= \mathbf{ZFC}^2) \\
\vdots \\
\vee \\
\mathbf{NBG} + \Pi^1_1\text{-}\mathbf{CA} \\
\vee \\
\mathbf{NBG{+}ETR} \\
\vee \\
\mathbf{ZF} \equiv \mathbf{ZFC} \equiv \mathbf{NBG} \\
\hline
\vee \\
\mathbf{Z} \\
\vee \\
\mathbf{Z}_{<\omega} \equiv \mathbf{ZBQC} \\
\vdots \\
\vee \\
\mathbf{Z}_3 \\
\vee \\
\mathbf{Z_2} \equiv \mathbf{ZFC{-}Pow}
\end{array}$$

$$\begin{array}{c}
\mathbf{ZFC} + \text{“}0 = 1\text{”} \\
\vdots \\
\vee \\
\mathbf{ZFC{+}Vop} \\
\vee \\
\mathbf{ZFC{+}SCpt} \\
\vee \\
\mathbf{ZFC{+}Wood} \\
\vee \\
\mathbf{ZFC{+}Meas} \\
\vee \\
\mathbf{ZFC} + 0^\sharp \\
\vee \\
\mathbf{ZFC{+}WCpt} \\
\vdots \\
\vee \\
\mathbf{ZFC} + 2\text{-}\mathbf{Mahlo} \\
\vee \\
\mathbf{ZFC{+}Mahlo} \\
\vdots \\
\vee \\
\mathbf{ZFC} + \omega\text{-}\mathbf{Inac} \\
\vdots \\
\vee \\
\mathbf{ZFC} + 2\text{-}\mathbf{Inac} \\
\vee \\
\mathbf{ZFC{+}Inac}
\end{array}$$

Picture of the hierarchy

$Z_2 \equiv ZFC{-}Pow$
$\vdots$
$\vee$
Π^1_2-CA_0
$\vee$
Σ^1_2-$AC \equiv KPi \equiv T_0$
$\vee$
$ID_{<\omega} \equiv \Pi^1_1$-$CA_0$
$\vee$
$ID_1 \equiv BI \equiv KP \equiv CZF \equiv MLT$
$\vee$
$\widehat{ID}_{<\omega} \equiv IR \equiv ATR_0$
$\vee$
$PA \equiv ACA_0 \equiv \Sigma^1_1$-$AC_0 \equiv HA$
$\vdots$
$\vee$
$I\Sigma_2$
$\vee$
$PRA \equiv I\Sigma_1 \equiv RCA_0 \equiv WKL_0$
$\vee$
$Q \equiv I\Sigma_0 + \Omega_1$

$ZFC{+}Inac$
$\vdots$
$\vee$
ZFC^3
$\vee$
$MK(:= ZFC^2)$
$\vdots$
$\vee$
$NBG{+}\Pi^1_1$-CA
$\vee$
$NBG{+}ETR$
$\vee$
$ZF \equiv ZFC \equiv NBG$
$\vee$
Z
$\vee$
$Z_{<\omega} \equiv ZBQC$
$\vdots$
$\vee$
Z_3
$\vee$
$Z_2 \equiv ZFC{-}Pow$

$ZFC{+}``0=1"$
$\vdots$
$\vee$
$ZFC{+}Vop$
$\vee$
$ZFC{+}SCpt$
$\vee$
$ZFC{+}Wood$
$\vee$
$ZFC{+}Meas$
$\vee$
$ZFC{+}0^\sharp$
$\vee$
$ZFC{+}WCpt$
$\vdots$
$\vee$
$ZFC{+}2$-$Mahlo$
$\vee$
$ZFC{+}Mahlo$
$\vdots$
$\vee$
$ZFC{+}\omega$-$Inac$
$\vdots$
$\vee$
$ZFC{+}2$-$Inac$
$\vee$
$ZFC{+}Inac$