Gödel's Theorem: Inconsistency vs Incompleteness

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November 4, 2016

Plan

Statement of the Theorem

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- *Inaccurate*: it must be either incomplete or inconsistent.

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B There is a formula with two free variables, $B(x, y)$, which represents the proof relation of T. That is:

  (i) if n is the code of a proof of A in T then $B(\mathbf{n}, \langle A \rangle)$ is true in the standard model
  (ii) if n is the not code of a proof of A in T then $\neg B(\mathbf{n}, \langle A \rangle)$ is true in the standard model

Assumptions Ctd.

C Define $Prov(y)$ as $\exists x B(x, y)$. Then $Prov$ is a proof predicate for T. That is:

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D There is a formula, G, of the form $\neg Prov\langle G \rangle$

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- If $T \vdash G$ then $T \vdash \neg Prov\langle G \rangle$.

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  - $\neg \exists x B(x, \langle G \rangle)$
  - $\neg Prov\,\langle G \rangle$
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- So if T is consistent, it is incomplete.
- Contrapositively, if T is complete, it is inconsistent.

Löb's Principle

- if $Prov \langle A \rangle$ then A

Naive Arithmetic

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- Let T be the theory containing all the things which are analytically true in this language.
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- Write $\vdash$ for provability in T.

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- The language contains a monadic predicate, P, which expresses this notion of provability in T.
  - [1] $\vdash \neg P\langle A\rangle \vee A$
  - [2] $\vdash A$ then $\vdash P\langle A\rangle$

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- Note: This does not show that the P-free fragment of T is inconsistent.

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- $T \vdash Prov \langle L \rangle \supset A$
- $T \vdash L$

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- $T \vdash L$
- $T \vdash Prov\,\langle L\rangle$

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Modelling Löbs Principle

- Let $\mathfrak{M}$ be a finite collapsed model of the standard model of arithmetic
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- Everything true in the standard model is in T
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- Let *Prov* be the arithmetic formula that defines T in the standard model

  [3] if $A \in T$, $Prov\,\langle A \rangle$ is true in the standard model, and so is in T

  [4] if $A \notin T$, $\neg Prov\,\langle A \rangle$ is true in the standard model, and so is in T

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- For $\vdash \neg P\langle A\rangle \vee A$:
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  - In the first case, $\neg Prov\langle A\rangle \vee A \in T$.
  - In the second case, $\neg Prov\langle A\rangle \in T$
  - So $\neg Prov\langle A\rangle \vee A \in T$.

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  - So $\neg Prov\langle A\rangle \vee A \in T$.

- Moreover, unless the collapsed model is one in which everything is identified with 0, T is non-trivial.

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- Is Naive Arithmetic axiomatizable?

- Learning how to prove things in arithmetic is a skill that is taught and learned.
- The assumption that the canons of naive proof are axiomatic is the most natural explanation of this fact.
- Any other explanation would make the grasp of the canons something of a mystery for human cognition.

Practical Consistency

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- n could be inordinately large

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- Then for every n, $\neg B(\mathbf{n}, \langle 0 = 1 \rangle)$ is true in the standard model
- So $\neg \exists x B(x, \langle 0 = 1 \rangle)$ is true in the standard model
- That is, $\neg Prov \langle 0 = 1 \rangle$ is true in the standard model.

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- So $\neg \exists x B(x, \langle 0 = 1 \rangle)$ is true in the standard model
- That is, $\neg Prov \langle 0 = 1 \rangle$ is true in the standard model.
- So $T \vdash \neg Prov \langle 0 = 1 \rangle$.

Fin