Introduction to Justification Logic

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Modal Logic

Modal logic adds a new connective $\Box$ to the language of logic.

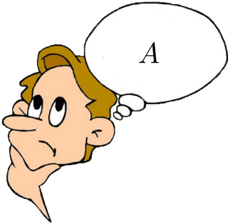
Two traditions:

Epistemic logic:

$\Box A$ means A is known / believed

Proof theory:

$\Box A$ means A is provable in system S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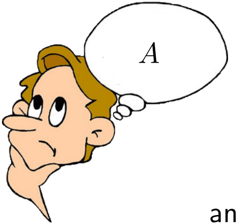
Modal Logic: How It Works



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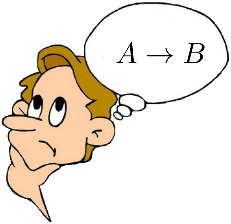

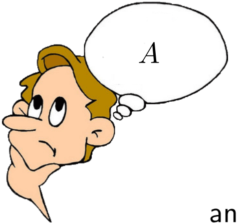

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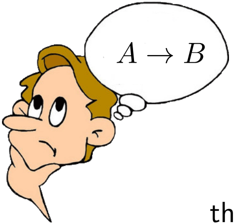

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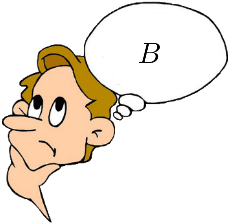

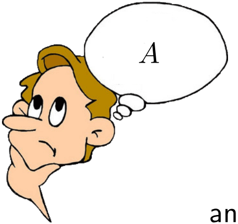

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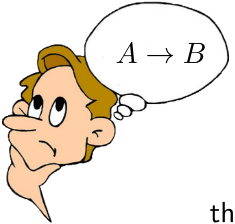

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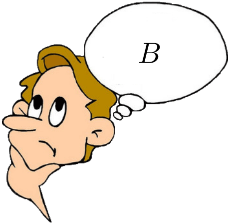
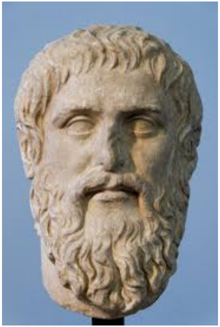


$$\Box A \quad \wedge \quad \Box(A \to B) \quad \to \quad \Box B$$

Problems: Epistemic Tradition



Plato:

Knowledge is justified true belief

True belief is modeled by $\Box A \to A$ but

where are the justifications in modal logic?

Problems: Proof-Theoretic Tradition

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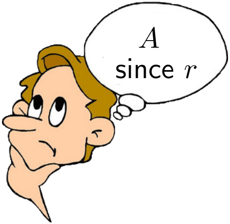
Gödel: if S has a certain strength, it cannot prove its own consistency.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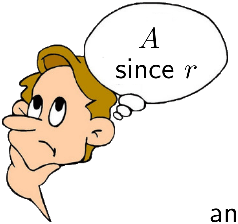
Justification Logic



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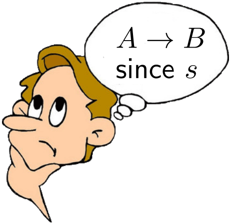
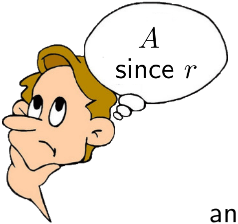


Justification Logic



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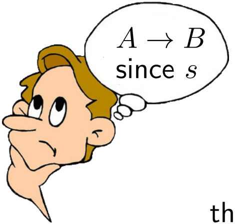

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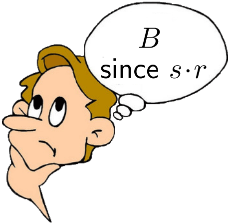
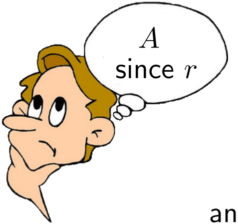


Justification Logic



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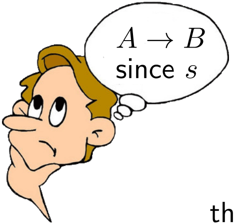

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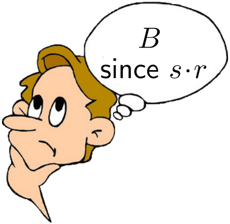

$$r{:}A \quad \wedge \quad s{:}(A \to B) \quad \to \quad s \cdot r{:}B$$

Syntax of the Logic of Proofs

Logic

The logic of proofs $\mathsf{LP}_{\mathsf{CS}}$ is the justification counterpart of the modal logic S4.

Justification terms Tm

$$t ::= x \mid c \mid (t \cdot t) \mid (t + t) \mid !t$$

Formulas $\mathcal{L}_j$

$$A ::= p \mid \neg A \mid (A \to A) \mid t{:}A$$

Axioms for LP

- all propositional tautologies
- $t{:}(A \to B) \to (s{:}A \to (t \cdot s){:}B)$ (J)
- $t{:}A \to (t + s){:}A, \quad s{:}A \to (t + s){:}A$ (+)
- $t{:}A \to A$ (jt)
- $t{:}A \to !t{:}t{:}A$ (j4)

Deductive System

Constant specification

A constant specification CS is any subset

$$\mathsf{CS} \subseteq \{(c, A) \mid c \text{ is a constant and } A \text{ is an axiom}\}.$$

The deductive system $\mathsf{LP}_{\mathsf{CS}}$ consists of the above axioms and the rules of modus ponens and axiom necessitation.

$$\frac{A \quad A \to B}{B} \qquad\qquad \frac{(c, A) \in \mathsf{CS}}{c{:}A}$$

A Justified Version of $\Box A \lor \Box B \to \Box(A \lor B)$

Assume we are given $\mathsf{LP}_{\mathsf{CS}}$ with

$$(a, A \to (A \lor B)) \in \mathsf{CS} \quad \text{and} \quad (b, B \to (A \lor B)) \in \mathsf{CS} \ .$$

By axiom necessitation we get

$$\mathsf{LP}_{\mathsf{CS}} \vdash a{:}(A \to (A \lor B)) \quad \text{and} \quad \mathsf{LP}_{\mathsf{CS}} \vdash b{:}(B \to (A \lor B)) \ .$$

Using (J) and (MP) we obtain

$$\mathsf{LP}_{\mathsf{CS}} \vdash x{:}A \to (a \cdot x){:}(A \lor B) \quad \text{and} \quad \mathsf{LP}_{\mathsf{CS}} \vdash y{:}B \to (b \cdot y){:}(A \lor B) \ .$$

Finally, from $(+)$ we have

$$\mathsf{LP}_{\mathsf{CS}} \vdash (a \cdot x){:}(A \lor B) \to (a \cdot x + b \cdot y){:}(A \lor B) \text{ and}$$
$$\mathsf{LP}_{\mathsf{CS}} \vdash (b \cdot y){:}(A \lor B) \to (a \cdot x + b \cdot y){:}(A \lor B) \ .$$

Using propositional reasoning, we obtain

$$\mathsf{LP}_{\mathsf{CS}} \vdash (x{:}A \lor y{:}B) \to (a \cdot x + b \cdot y){:}(A \lor B) \ .$$

Internalization

Definition

A constant specification CS for LP is called *axiomatically appropriate* if for each axiom F of LP, there is a constant c such that $(c, F) \in \mathsf{CS}$.

Lemma (Internalization)

Let CS be an axiomatically appropriate constant specification. For arbitrary formulas $A, B_1, \ldots, B_n$*, if*

$$B_1, \ldots, B_n \vdash_{\mathsf{LP}_{\mathsf{CS}}} A \quad ,$$

then there is a term t *such that*

$$x_1{:}B_1, \ldots, x_n{:}B_n \vdash_{\mathsf{LP}_{\mathsf{CS}}} t{:}A$$

for fresh variables $x_1, \ldots, x_n$*.*

Forgetful Projection

Definition (Forgetful projection)

The mapping $\circ$ from justified formulas to modal formulas is defined as follows

1. $P^\circ := P$ for P atomic;
2. $(\neg A)^\circ := \neg A^\circ$;
3. $(A \to B)^\circ := A^\circ \to B^\circ$;
4. $(t{:}A)^\circ := \Box A^\circ$.

Lemma (Forgetful projection)

For any constant specification CS *and any formula* F *we have*

$$\mathsf{LP}_{\mathsf{CS}} \vdash F \quad \textit{implies} \quad \mathsf{S4} \vdash F^\circ \ .$$

Realization

Definition (Realization)

A *realization* is a mapping r from modal formulas to justified formulas such that $(r(A))^\circ = A$.

Definition

We say a justification logic $\mathsf{LP}_{\mathsf{CS}}$ *realizes* S4 if there is a realization r such that for any formula A we have

$$\mathsf{S4} \vdash A \quad \text{implies} \quad \mathsf{LP}_{\mathsf{CS}} \vdash r(A) \ .$$

Realization Theorem

Definition (Schematic CS)

We say that a constant specification is *schematic* if it satisfies the following: for each constant c, the set of axioms $\{A \mid (c, A) \in \mathsf{CS}\}$ consists of all instances of one or several (possibly zero) axiom schemes of LP.

Theorem (Realization)

Let CS be an axiomatically appropriate and schematic constant specification. There exists a realization r such that for all formulas A

$$\mathsf{S4} \vdash A \quad \Longrightarrow \quad \mathsf{LP}_{\mathsf{CS}} \vdash r(A) \ .$$

Arithmetical Semantics

Originally, LP_{CS} was developed to provide classical provability semantics for intuitionistic logic.

Arithmetical Semantics for LP_{CS}: Justification terms are interpreted as proofs in Peano arithmetic and operations on terms correspond to computable operations on proofs in PA.

$$\mathsf{Int} \quad \xrightarrow{\text{Gödel}} \quad \mathsf{S4} \quad \xrightarrow{\text{Realization}} \quad \mathsf{JL} \quad \xrightarrow{\text{Arithm. sem.}} \quad \mathsf{CL} + \text{proofs}$$

Self-referentiality

Definition (Self-referential CS)

A constant specification CS is called *self-referential* if $(c, A) \in$ CS for some axiom A that contains at least one occurrence of the constant c.

S4 and $\mathsf{LP}_{\mathsf{CS}}$ describe self-referential knowledge. That means if $\mathsf{LP}_{\mathsf{CS}}$ realizes S4 for some constant specification CS, then that constant specification must be self-referential.

Lemma

Consider the S4*-theorem* $G := \neg\Box((P \to \Box P) \to \bot)$ *and let* F *be any realization of* G*.*
If $\mathsf{LP}_{\mathsf{CS}} \vdash F$*, then* CS *must be self-referential.*

Basic evaluation

Definition (Basic Evaluation)

A *basic evaluation* $*$ for $\mathsf{LP}_{\mathsf{CS}}$ is a function:

$$*: \mathsf{Prop} \to \{0,1\} \text{ and } *: \mathsf{Tm} \to \mathcal{P}(\mathcal{L}_j), \text{ such that}$$

1. $F \in (s \cdot t)^*$ if $(G \to F) \in s^*$ and $G \in t^*$ for some G
2. $F \in (s + t)^*$ if $F \in s^*$ or $F \in t^*$
3. $F \in t^*$ if $(t, F) \in \mathsf{CS}$
4. $s{:}F \in (!s)^*$ if $F \in s^*$

Quasimodel

Definition (Quasimodel)

A *quasimodel* is a tuple $\mathcal{M} = (W, R, *)$ where $W \neq \varnothing$, $R \subseteq W \times W$, and the *evaluation* $*$ maps each world $w \in W$ to a basic evaluation $*_w$.

Definition (Truth in quasimodels)

$$\mathcal{M}, w \Vdash p \text{ if and only if } p^*_w = 1 \text{ for } p \in \mathsf{Prop};$$
$$\mathcal{M}, w \Vdash F \to G \text{ if and only if } \mathcal{M}, w \nVdash F \text{ or } \mathcal{M}, w \Vdash G;$$
$$\mathcal{M}, w \Vdash \neg F \text{ if and only if } \mathcal{M}, w \nVdash F;$$
$$\mathcal{M}, w \Vdash t{:}F \text{ if and only if } F \in t^*_w.$$

Model

Given $\mathcal{M} = (W, R, *)$ and $w \in W$, we define

$$\Box_w := \{F \in \mathcal{L}_j \mid \mathcal{M}, v \Vdash F \text{ whenever } R(w, v)\} \ .$$

Definition (Modular Model)

A *modular model* $\mathcal{M} = (W, R, *)$ is a quasimodel with

1. $t^*_w \subseteq \Box_w$ for all $t \in \mathsf{Tm}$ and $w \in W$; (JYB)
2. R is reflexive;
3. R is transitive.

Theorem (Soundness and Completeness)

For all formulas $F \in \mathcal{L}_j$,

$$\mathsf{LP}_{\mathsf{CS}} \vdash F \qquad \Longleftrightarrow \qquad \mathcal{M} \Vdash F \textit{ for all modular models } \mathcal{M}.$$

Decidability

In modal logic, decidability is a consequence of the finite model property. For $\mathsf{LP}_{\mathsf{CS}}$ the situation is more complicated since CS usually is infinite.

Theorem

$\mathsf{LP}_{\mathsf{CS}}$ *is decidable for decidable schematic constant specifications* CS.

A decidable CS is not enough:

Theorem

There exists a decidable constant specification CS *such that* $\mathsf{LP}_{\mathsf{CS}}$ *is undecidable.*

Complexity

Theorem

Let CS *be a schematic constant specification.*
The problem whether $\mathsf{LP}_{\mathsf{CS}} \vdash t{:}B$ *belongs to NP.*

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Definition

A constant specification is called *schematically injective* if it is schematic and each constant justifies no more that one axiom scheme.

Theorem

Let CS *be a schematically injective and axiomatically appropriate constant specification.*
The derivability problem for $\mathsf{LP}_{\mathsf{CS}}$ *is* Π_2^p*-complete.*

Thank you!