MANY-VALUED MODAL LOGICS

Denisa Diaconescu – University of Bern, University of Bucharest

SGSLPS Meeting
October 30th, 2015

WHAT ARE MANY-VALUED MODAL LOGICS?

Modal Logic Modal Logic Modal Logic Modal Logic Modal Logic

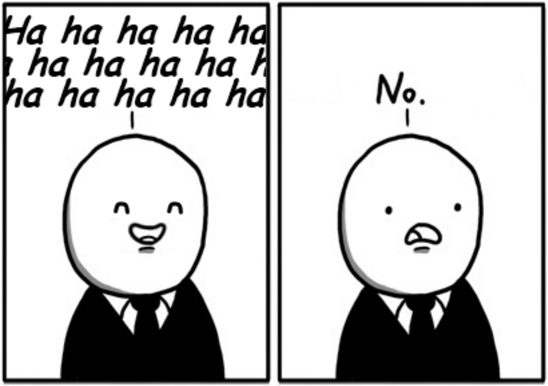
Modal Logic dal Logic

Modal Logic dal Logic

Modal Logic dal Logic

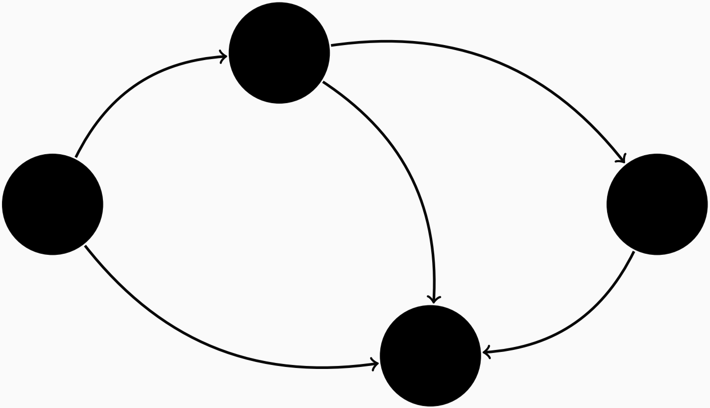
Modal Logic dal Logic

Modal Logic dal Logic

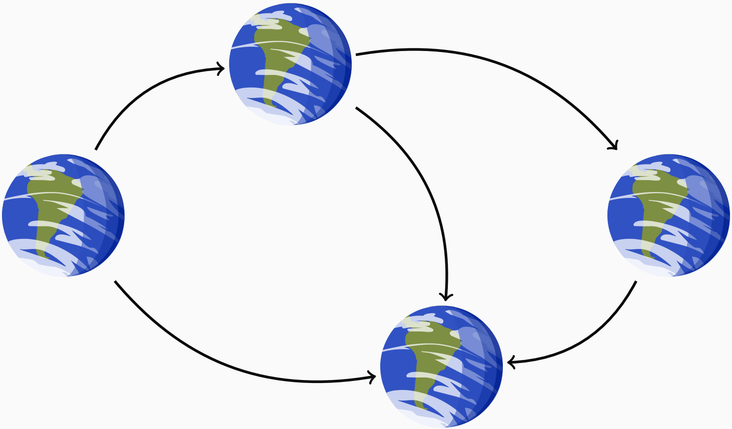


Modal Logic Modal Logic Modal Logic Modal Logic Modal Logic

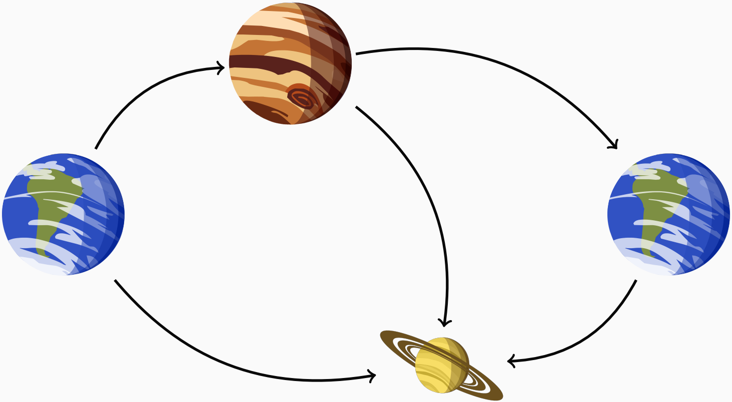
POSSIBLE WORLDS SEMANTICS



MANY-VALUED WORLDS

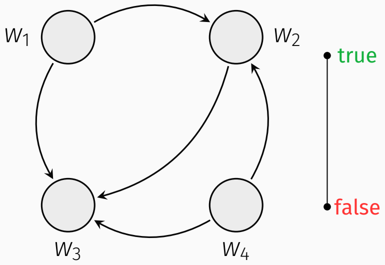


MANY-VALUED WORLDS



Kripke model:

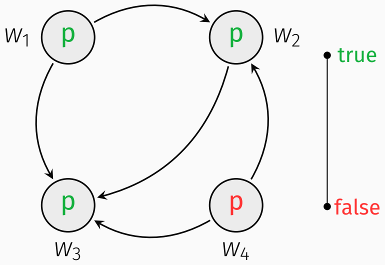
- Kripke frame: $\langle W, R \rangle$
  $R \subseteq W \times W$
- Valuation:
  $V : Var \times W \to \{\text{false}, \text{true}\}$



FIRST EXAMPLE

Kripke model:

- Kripke frame: $\langle W, R\rangle$
  $R \subseteq W \times W$
- Valuation:
  $V : Var \times W \to \{\text{false}, \text{true}\}$



Extending V:

$V(\Box\varphi, w) = \bigwedge_{R(w,w')} V(\varphi, w')$

$V(\Box p, w_1) = \text{true}$

What happens if V is allowed to be partial?

A variable at a world can be true, false or undefined!

How can we extend the incomplete information in V to all formulas?

- $\varphi \wedge \psi$ at a world w should be
  - true if both φ and ψ are true at w
  - false if one of φ or ψ is false at w
  - undefined in all other situations
- $\Box\varphi$ at a world w should be
  - true if for all w' with $R(w, w')$, φ is true at w',
  - false if there is some w' with $R(w, w')$ such that φ is false at w'
  - undefined in all other situations

FIRST EXAMPLE

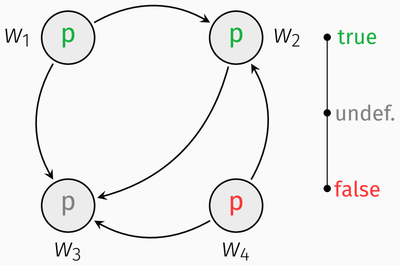
Three-valued logic

$\wedge$	f	u	t
f	f	f	f
u	f	u	u
t	f	u	t

FIRST EXAMPLE

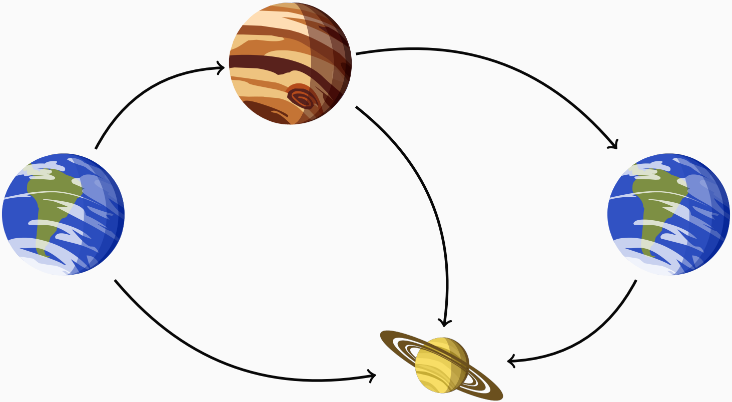
Kripke model:

- Kripke frame: $\langle W, R \rangle$
  $R \subseteq W \times W$
- Valuation:
  $V : Var \times W \to \{\text{false}, \text{undefined}, \text{true}\}$

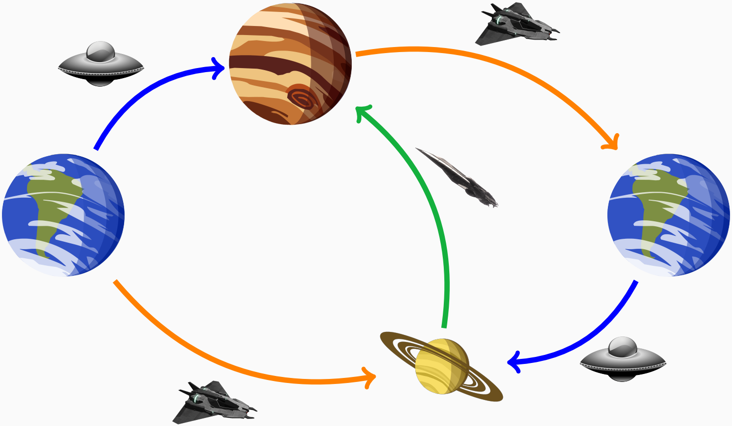


$$V(\Box\varphi, w) = \bigwedge_{R(w,w')} V(\varphi, w')$$

$$V(\Box p, w_1) = \text{undefined}$$



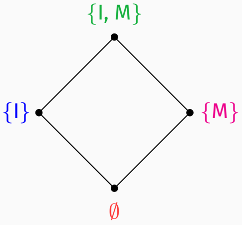
MANY-VALUED ACCESSIBILITY RELATION



Suppose we have two experts **Ion** – **I** and **Maria** – **M** who are being asked to pass judgement on the truth of various statements, in various situations.

The truth-valued space is a four-valued one:

- neither says true
- **I** says true, but **M** says no
- **M** says true, but **I** says no
- both says yes

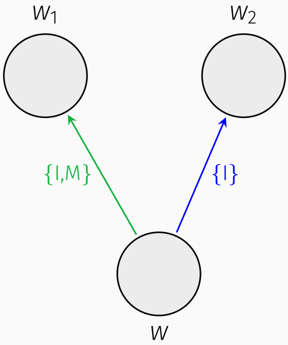


Two kinds of judgements are possible:

- the statement φ is true in the situation w
- w is a situation that should be considered

Consider the scenario:

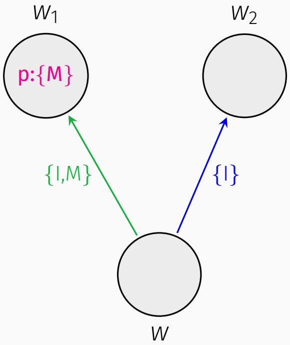
- Both I and M say w_1 should be considered
- Only I says w_2 should be considered



SECOND EXAMPLE

Consider the scenario:

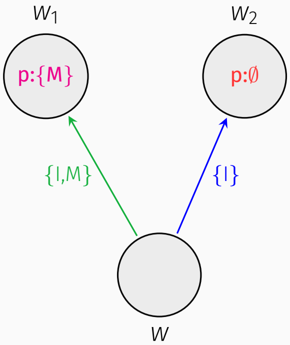
- Both I and M say w_1 should be considered
- Only I says w_2 should be considered
- Only M says p would be true in situation w_1



Consider the scenario:

- Both I and M say w_1 should be considered
- Only I says w_2 should be considered

- Only M says p would be true in situation w_1
- Nobody says p would be true in situation w_2



How should $\Box p$ be evaluated in world w?

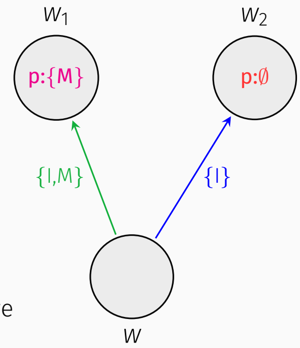
In a sense, it should be what is common to all alternative situations.

For example, $V(\Box p, w) = V(p, w_1) \wedge V(p, w_2) = \emptyset$.

SECOND EXAMPLE

We must also take into account which situations should be considered!

- For w_1:
  - Everybody says it should be considered!
  - From **I** we get a no.
  - From **M** we get a yes.
  - Thus w_1 contributes {**M**}.
- For w_2:
  - **I** says w_2 should be considered and that p is false there, so from **I** we get a no.
  - **M** does not say it should be considered at all, so we count from **M** a yes.
  - Thus w_2 contributes {**M**}.



Therefore $V(\Box p, w) = \{\mathbf{M}\}$

On a closer examination, we used the following rule:

The truth value of $\Box\varphi$ is the intersection, over all worlds,
of the truth value of φ at an alternative world
union the complement of the accessibility value of that alternative world.

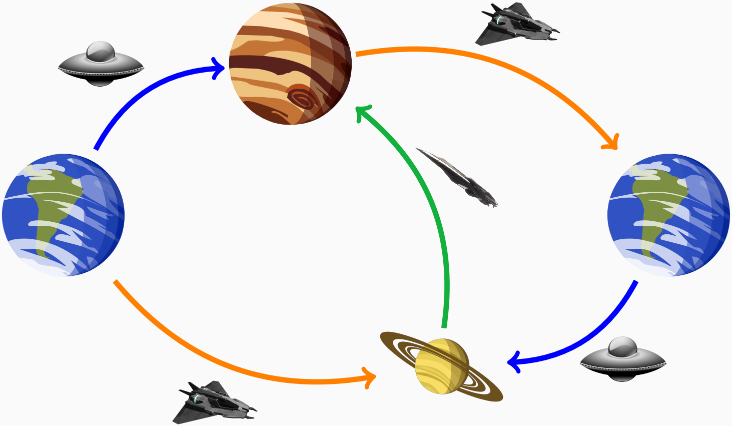
$$V(\Box\varphi, w) = \bigwedge\{\bar{R}(w, w') \vee V(\varphi, w') \ : \ \text{all } w'\}$$

On a closer examination, we used the following rule:

The truth value of $\Box\varphi$ is the intersection, over all worlds,
of the truth value of φ at an alternative world
union the complement of the accessibility value of that alternative world.

$$V(\Box\varphi, w) = \bigwedge\{R(w, w') \to V(\varphi, w') \ : \ \text{all } w'\}$$

(Fitting)



LET’S GET FORMAL

Kripke model:

- Kripke frame: $\langle W, R \rangle$
  $R \subseteq W \times W$
- Valuation:
  $V : Var \times W \to \{\text{false}, \text{true}\}$

Kripke model:

- Kripke frame: $\langle W, R \rangle$
  $R : W \times W \rightarrow \{0, 1\}$
- Valuation:
  $V : Var \times W \rightarrow \{0, 1\}$

Kripke model:

- Kripke frame: $\langle W, R \rangle$

  $R : W \times W \to ?$

- Valuation:

  $V : Var \times W \to ?$

A residuated lattice is a structure $\boldsymbol{A} = \langle A, \wedge, \vee, \odot, \to, 0, 1\rangle$ such that:

- $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice with top 1 and bottom 0
- $\langle A, \odot, 1\rangle$ is a commutative monoid
- $\to$ is the residuum of the $\odot$, i.e.

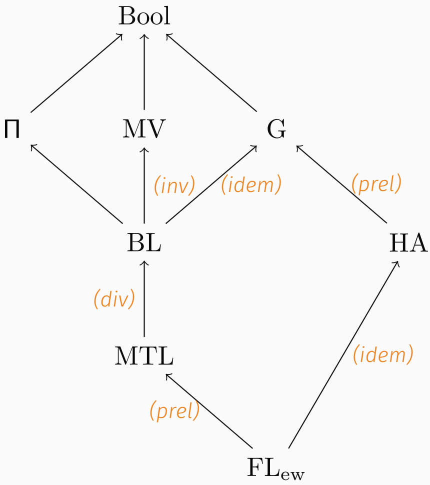
$$x \odot y \leq z \quad \Leftrightarrow \quad x \leq y \to z \qquad \text{for all } x, y, z \in A$$

integral, commutative residuated monoids

$\mathrm{FL_{ew}}$-algebras

To any residuated lattice $\boldsymbol{A}$ there is a natural way to associate a logic $\mathrm{Log}(\boldsymbol{A})$.

RESIDUATED LATTICES AND SUBVARIETIES



(idem) $x \odot x = x$

(prel) $(x \to y) \vee (y \to x) = 1$

(div) $x \wedge y = x \odot (x \to y)$

(inv) $\neg\neg x = x$

A residuated lattice $\mathbf{A}$ is complete if

$$\bigvee X \quad \text{and} \quad \bigwedge X \quad \text{exist in } A \quad \text{for all } X \subseteq A.$$

Example

- Standard Łukasiewicz algebra $[0,1]_{Ł}$

$$\begin{aligned} x \odot y &= \max\{0\,, x + y - 1\} \\ x \to y &= \min\{1\,, 1 - x + y\} \end{aligned}$$

- Standard Gödel algebra $[0,1]_{G}$

$$\begin{aligned} x \odot y &= x \wedge y \\ x \to y &= \begin{cases} 1\,, & \text{if } x \leq y, \\ 0\,, & \text{if } y < x \end{cases} \end{aligned}$$

Let A be a complete residuated lattice.

An (A-valued) Kripke frame is a pair $\mathfrak{F} = \langle W, R \rangle$ where

$R : W \times W \to A$

A kripke frame $\mathfrak{F} = \langle W, R \rangle$ is called

- crisp (or classical) if $R[W \times W] \subseteq \{0,\ 1\}$
- idempotent if $R[W \times W] \subseteq \{a \in A\ :\ a \odot a = a\}$

$\mathbf{CFr}$	$\subseteq$	$\mathbf{IFr}$	$\subseteq$	$\mathbf{Fr}$
crisp Kripke frames		idempotent Kripke frames		Kripke frames

An ($\boldsymbol{A}$-valued) Kripke model is a pair $\mathfrak{M} = \langle W, R, V \rangle$ where

- $\mathfrak{F} = \langle W, R \rangle$ is an ($\boldsymbol{A}$-valued) Kripke frame
- $V : Var \times W \to \boldsymbol{A}$ is a valuation

We can extend to $V : Fm \times W \to \boldsymbol{A}$ by

- $V(\varphi \circ \psi, w) = V(\varphi, w) \circ V(\psi, w)$ where $\circ \in \{\wedge, \vee, \odot, \to\}$
- $V(\Box\varphi, w) = \bigwedge\{R(w, w') \to V(\varphi, w') \ : \ w' \in W\}$

$$V(\Diamond\varphi, V) = \bigvee\{R(w, w') \odot V(\varphi, w') : w' \in W\}$$

In general, we cannot define $\Diamond$ as an abbreviation of $\neg\Box\neg$!

We can do this without troubles in the involutive cases.

If $\mathfrak{M} = \langle W, R, V \rangle$ is a Kripke model and $w \in W$,
$\mathfrak{F} = \langle W, R \rangle$ is a Kripke frame, and $\mathbb{K}$ is a class of Kripke frames,
we

write	say	if
$\mathfrak{M}, w \models^1 \varphi$ $w \models^1 \varphi$	w validates φ	$V(\varphi, w) = 1$
$\mathfrak{M} \models^1 \varphi$	φ is valid in $\mathfrak{M}$	$w \models^1 \varphi$, for every $w \in W$
$\mathfrak{F} \models^1 \varphi$	φ is valid in $\mathfrak{F}$	φ is valid in any Kripke model based on $\mathfrak{F}$
$\mathbb{K} \models^1 \varphi$		φ is valid in all frames in $\mathbb{K}$

THE NORMALITY AXIOM

In general, the normality axiom (K) is not valid in Fr!

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \qquad \text{(K)}$$

Example

$V(\Box(p \to q), w)$

$= R(w,w) \to V(p \to q, w)$

$= 0.5 \to 0.5 = 1$

$V(\Box p, w) = R(w,w) \to V(p,w)$

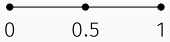
$= 0.5 \to 0.5 = 1$

$V(\Box q, w) = 0.5 \to 0 = 0.5$

$V(\Box(p \to q) \to (\Box p \to \Box q), w)$

$= 1 \to (1 \to 0.5) = 0.5$

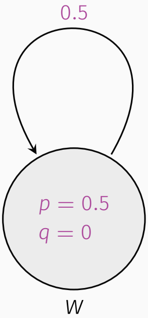
MV-chain



$x \odot y = \max\{0\,,\, x + y - 1\}$

$x \to y = \min\{1\,,\, 1 - x + y\}$

Kripke model



Theorem (Bou – Esteva – Godo – Rodríguez)

- *Some valid formulas in* FR *are*

$$(\Box\varphi \wedge \Box\psi) \leftrightarrow \Box(\varphi \wedge \psi)$$
$$\neg\neg\Box\varphi \to \Box\neg\neg\varphi$$

- *Some valid formulas in* IFR *are*

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$
$$(\Box\varphi \odot \Box\psi) \to \Box(\varphi \odot \psi)$$

- *Some valid formulas in* CFR *are*

$$\Box 0 \vee \neg\Box 0$$

Theorem (Bou – Esteva – Godo – Rodríguez)

Axiom (K) is valid in $\mathbf{Fr}$ iff A is a Heyting algebra
iff $\mathbf{Fr} = \mathbf{IFr}$

Let us remark two particular cases when axiom (K) holds:

- when $\odot$ and $\wedge$ coincide
- in all crisp Kripke frames $\mathbf{CFr}$

MANY-VALUED MODAL LOGICS

What is the many-valued counterpart of the minimum classical modal logic K?

Let A be a complete residuated lattice and $\mathbb{F}$ be a class of Kripke frames.

The many-valued modal logic $\mathsf{Log}_\Box(A, \mathbb{F})$ is defined as the set of formulas $\varphi \in Fm_\Box$ satisfying that

for every A-valued Kripke model $\mathfrak{M}$ over a frame in $\mathbb{F}$, $\mathfrak{M} \models^1 \varphi$.

How can we axiomatize the minimal logic $\mathsf{Log}_\Box(A, \mathrm{Fr})$*?*

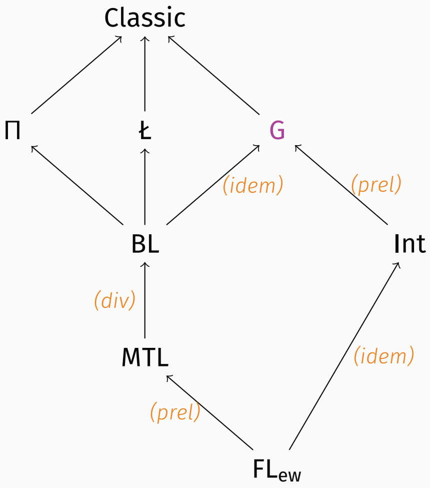
What axioms and rules must we add to
an axiomatization of $\mathsf{Log}(A)$ *to get an axiomatization of* $\mathsf{Log}_\Box(A, \mathbb{F})$*?*

Let A be a complete residuated lattice and $\mathbb{F}$ be a class of Kripke frames.

The many-valued modal consequence $\models_{\Box(A,\mathbb{F})}$ is defined by

$\Gamma \models_{\Box(A,\mathbb{F})} \varphi$ iff for every A-valued Kripke model $\mathfrak{M}$ over a frame in $\mathbb{F}$, if $\mathfrak{M} \models^1 \Gamma$, then $\mathfrak{M} \models^1 \varphi$.

The set of theorems of $\models_{\Box(A,\mathbb{F})}$ is precisely the set $\mathbf{Log}_{\Box}(A, \mathbb{F})$.



Standard Gödel algebra $[0,1]_G$

$$x \odot y = x \wedge y$$

$$x \to y = \begin{cases} 1\,, & \text{if } x \leq y, \\ 0\,, & \text{if } y < x \end{cases}$$

$\mathbf{Log}_{\Box}([0,1]_{\mathsf{G}}, \mathrm{Fr})$ is axiomatized by the axioms of $\mathbf{Log}([0,1]_G)$ and

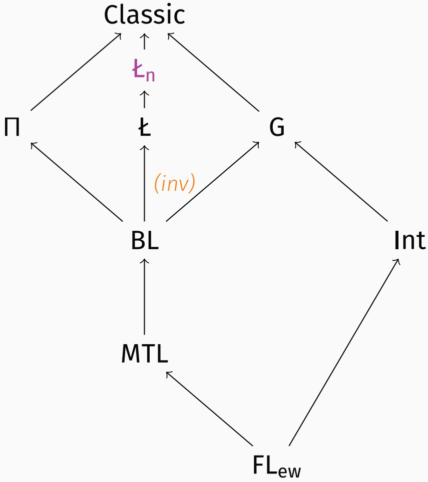
$$(K) \quad \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

$$(Z) \quad \neg\neg\Box\varphi \to \Box\neg\neg\varphi$$

and has the Modus Ponens rule and the Necessity rule.

Moreover, $\mathbf{Log}_{\Box}([0,1]_{\mathsf{G}}, \mathrm{Fr}) = \mathbf{Log}_{\Box}([0,1]_{\mathsf{G}}, \mathrm{CFr})$

(Caicedo – Rodríguez,
Metcalfe – Olivetti)



For any strictly positive integer n,

$$Ł_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$$

$$x \odot y = \max\{0, x + y - 1\}$$

$$x \to y = \min\{1, 1 - x + y\}$$

$\mathsf{Log}_{\Box}(\text{Ł}_n, \mathrm{CFr})$ is axiomatized by the axioms of $\mathsf{Log}(\text{Ł}_n)$ and

$$(K) \quad \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

$$\Box(\varphi \oplus \varphi) \leftrightarrow \Box\varphi \oplus \Box\varphi$$

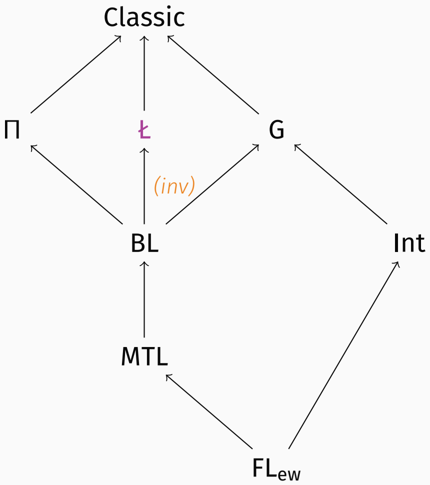
$$\Box(\varphi \odot \varphi) \leftrightarrow \Box\varphi \odot \Box\varphi$$

and has the Modus Ponens rule and the Necessity rule.

(Hansoul – Teheux)

An axiomatiozation for $\mathsf{Log}_{\Box}(\text{Ł}_n, \mathrm{Fr})$ is also known.

(Bou – Esteva – Godo – Rodríguez)



Standard Łukasiewicz algebra

$[0,1]_Ł$

$$x \odot y = \max\{0, x+y-1\}$$

$$x \to y = \min\{1, 1-x+y\}$$

$\mathsf{Log}_{\Box}([0,1]_Ł, \mathrm{CFr})$ is axiomatized by the axioms of $\mathsf{Log}([0,1]_Ł)$ and

$$(K) \quad \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

$$\Box(\varphi \oplus \varphi) \leftrightarrow \Box\varphi \oplus \Box\varphi$$

$$\Box(\varphi \odot \varphi) \leftrightarrow \Box\varphi \odot \Box\varphi$$

$$\Box(\varphi \oplus \varphi^n) \leftrightarrow ((\Box\varphi) \oplus (\Box\varphi)^n)$$

and has the Modus Ponens rule, the Necessity rule and the infinitary rule

$$\frac{\varphi \oplus \varphi\ ,\ \varphi \oplus \varphi^2\ ,\ \ldots, \varphi \oplus \varphi^n\ , \ldots}{\varphi}$$

(Hansoul – Teheux)

A LOT OF QUESTIONS IN SEARCH OF AN ANSWER

- Can we avoid the infinitary rule for $\mathsf{Log}_{\Box}([0,1]_{Ł}, \mathrm{CFr})$?
- What about $\mathsf{Log}_{\Box}([0,1]_{Ł}, \mathrm{Fr})$?
- Axiomatizations for other cases when (K) fails?
- ...

Thank you for your attention!

All math is applied math... eventually.

Some metalogical properties are lost.

- If $\boldsymbol{A}$ and $\boldsymbol{B}$ generate the same variety, does not mean that $\mathbf{Log}_\Box(\boldsymbol{A}, \mathrm{CFr}) = \mathbf{Log}_\Box(\boldsymbol{B}, \mathrm{CFr})$!
  - $\Box\neg\neg p \to \neg\neg\Box p \notin \mathbf{Log}_\Box([0,1]_G, \mathrm{CFr})$
  - $\Box\neg\neg p \to \neg\neg\Box p \in \mathbf{Log}_\Box(\{0\} \cup [\frac{1}{2}, 1], \mathrm{CFr})$

  In general, the modal logic given by $\boldsymbol{A}$ does not coincide with the modal logic given by the variety generated by $\boldsymbol{A}$.

- It can happen that two classes F_1 and F_2 of crisp Kripke frames have different many-valued modal logics for an algebra $\boldsymbol{A}$, while for the case of the Boolean algebra of two elements they share the same logic.
  - F_1 the class of finite quasi-orders and F_2 the class of infinite partial orders
  - both F_1 and F_2 generates S_4
  - $\Box\neg\neg p \to \neg\neg\Box p \in \mathbf{Log}_\Box([0,1]_G, \mathrm{F}_1)$
  - $\Box\neg\neg p \to \neg\neg\Box p \notin \mathbf{Log}_\Box([0,1]_G, \mathrm{F}_2)$